UNIVERSITÉ D'ANTANANARIVO

DOMAINE SCIENCES ET TECHNOLOGIE

MENTION MATHÉMATIQUES ET INFORMATIQUE

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET APPLICATIONS

ÉQUIPE D'ACCUEIL DE MATHÉMATIQUES ET STRUCTURES

LABORATOIRE D’ALGÈBRE ET GÉOMÉTRIE

ENSEMBLE DE TRAVAUX DE RECHERCHES

DE

Princy RANDRIAMBOLOLONDRANTOMALALA

EN VUE DE L’OBTENTION DE L'

HABILITATION A DIRIGER DES RECHERCHES

DE

MATHÉMATIQUES

21 Juillet 2016

Cet ensemble de travaux est reparti en plusieurs petits ensembles selon la nature de chaque article qui le compose.

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2010 ***Université d'Antananarivo***: « **Doctorat de Mathématiques»**

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Encadreur: Pr. Anona; Mémoire '' Sur les algèbres de Lie d'un sous-module de champs de vecteurs sur l'anneau des fonctions ''

2000-2001 ***Université d'Antananarivo***: "Maîtrise de Mathématiques pures"

1999-2000 ***Université d'Antananarivo***: "Licence de Mathématiques"

1997 **"Baccalauréat, Série C" (Scientifique)** (Madagascar)

Recherches

Principaux Intérêts:

- **Primaire**: m-dérivations des algèbres de Lie de champs de vecteurs**,** m-dérivations des champs de vecteurs analytiques, Connexions (homogènes ou non homogènes) et les algèbres de Lie correspondantes, Espace de Nullité de courbure d'une connexion, Algèbres de Lie de champs de vecteurs et leurs propriétés algébriques, Feuilletages généralisés et leurs algèbres de Lie, Cohomologie de Chevalley-Eilenberg, m-dérivations des algèbres de Lie en Biomathématiques.

- **Secondaire**: Relations entre algèbres de Lie et groupes de Lie, Espaces de Finsler, Espaces de Riemann, Groupes d'Holonomies et groupes de Lie de transformations.

Editions scientifiques et Articles:

- Auteur de l'Edition Spéciale ''Recent Advances of Lie Theory in Differential Geometry, in Memory of John Nash, Journal of Generalized Lie theory and applications, 2016.

- Auteur de « Les m-dérivations des algèbres de Lie de champs de vecteurs polynomiaux», Vol 27, Afrika Matematika, 2016.

-Auteur de « A meeting of Great Minds, Sophus Lie and John Nash throughout their works», Special issue ''Recent Advances of Lie Theory in Differential Geometry, in Memory of John Nash, Journal of Generalized Lie theory and applications, S2: e001, 2015.

- Auteur de « The m-derivations of analytic vector fields Lie algebras », Special issue ''Recent Advances of Lie Theory in Differential Geometry, in Memory of John Nash, Journal of Generalized Lie theory and applications, S2: 002, 2015.

- Auteur de « The m-derivations of distribution Lie algebras », Journal of Generalized Lie theory and applications, 9: 1, 2015, doi : 10.4172/1736-4337.1000217.

- Premier auteur de « Sur les algèbres de Lie associées à une connexion », Canadian Mathematical Bulletin, 58 (n°4), pp. 692-703, 2015.

- Auteur de « Un petit résumé sur les maths pour les sciences vivantes et de la terre », Support de Cours Université d'Antananarivo, 2014.

- Auteur de « Un petit résumé sur l'optimisation linéaire », Support de Cours IT University, Madagascar (2014).

- Second auteur de « Sur les algèbres de Lie d'un système des champs de vecteurs permutables », Italian Journal of Pure and Applied Mathematics, 29, pp.163-174 (2012).

- Second auteur de « Sur les algèbres de Lie des champs de vecteurs polynomiaux », African Diaspora Journal of Mathematics, V. 10, number 2, pp.87-95 (2010).

- Premier auteur de « Sur les algèbres de Lie d'une distribution et d'un feuilletage généralisé », African Diaspora Journal of Mathematics, V. 10, number 2, pp.135-144 (2010).

Communications, Ecoles, Visites scientifiques:

- Exposé sur ''Les algèbres génétiques et leurs dérivations'', Salon de la Recherche Université d'Antananarivo 15-16 Octobre 2015.
- Exposé sur ''Les différents domaines de recherches dans le laboratoire d'algèbre et géométrie de l'Université d'Antananarivo'', Salon de la Recherche Université d'Antananarivo 15-16 Octobre 2015.
- Conférence ''Les champs de vecteurs polynomiaux sur les variétés analytiques'', Faculté des Sciences Université de Fianarantsoa, 17 Juin 2015.
- Conférence ''Le laboratoire d'algèbre et géométrie'', Faculté des Sciences Université de Fianarantsoa, 17 Juin 2015.

- Assistant au ''Recent progress in lagrangian and hamiltonian dynamics'' à l'ENS Lyon in honnor of 70th John Norman Mather's birthday , 4 Juin au 9 Juin 2012.
- Participant au séminaire ''Les systèmes d'Anosov'' CIRM (Marseille) France du 18 au 22 Juin 2012.
- Éditeur au Journal of Generalized Lie Theory pour une édition spéciale "Recent Progress of Lie Theory in Differential Geometry, in Memory of John Nash".
- Membre de la Comité d'organisateurs locaux et communicateur à "École de Combinatoire et algorithmique (ECAM) à l'Université d'Antananarivo (Madagascar) 6-14 Septembre 2012.
- Organisateur Local et participant à l'EMA (École Mathématiques Africaine) à l'Université d'Antananarivo (Madagascar) 19-30 Septembre 2011.
- Visite prévue au Laboratoire de Géométrie et Théorie des Algorithmes dirigé par Pr. Pierre Lecomte, Université de Liège Belgique.
- Visitant Pr. Abdelghani Zeghib de l'Unité de Mathématiques Pures et Appliquées (ENS Lyon) France du 24 Avril au 30 Juin 2012.

Expériences d'encadrement

Encadreur de DEA et Master 2 à l'Université d'Antananarivo (Géométrie Différentielle)

- Randriatsarafara Iharintsoa Hobiniaina ''Connexions et espace de nullité de la courbure" 16 Décembre 2014.

- **Lantonirina Sendrasoa Laurence ''**Sur les variétés riemanniennes'' 30 Avril 2015.

- Fahad Ben Abdou ''Les formes vectorielles et connexions de Grifone appliquées aux variétés finsleriennes à courbure régulière", 30 Avril 2015.

- **Randriamidosy Harimanana Roger** ''Sur les variétés analytiques" 11 Février 2016.

Activités pédagogiques :

- Acteur Pédagogique de la Mention Biologie Faculté des Sciences-Université d'Antananarivo pour le projet PGI Cocktail MADAGASCAR (2015).

- Deux fois Président du Jury des examens de la première année filière mathématiques à l'Université d'Antananarivo.

Expériences d'Enseignement

Master Département de Mathématiques et Informatique (Université d'Antananarivo)

Cours Master 1(Maths et Structures), Variétés différentielles 2014-2015
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Cours Master 1 (Mathématiques appliquées), Analyse Mathématique 2014-2015
Course Master 1 (Maths et Structures), Homologie et Cohomologie 2014-2015
Cours Master 2 (Maths et Structures), Variétés Riemanniennes 2014-2015
Assistant Master 2. Géométrie différentielle: 2011-2013

Licence Département de Mathématiques et Informatique (Université d'Antananarivo)

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3ème année Calcul Différentiel: 2008-2009
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Licence au Département de Sciences de la Vie et de la Terre (Université d'Antananarivo)

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Cours Licence 1 (Biologie), Maths pour SVT 2015

Licence aux autres Universités

Cours d'Outils mathématiques (Université ACEEM Antananarivo) 2007-2015
Cours de Probabilité et Statistique, Mathématiques Appliquées (ISIME Antananarivo) 2009-2011
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Membre fondateur et de la Société Mathématiques Malagasy (SMMG)
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Références

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Autres Qualifications

- Calculant sur Maple
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Anglais Très bien pour écrire et parler

Dernière mise à jour Mars 2016

Les m-derivations des algebres de Lie de champs de vecteurs polynomiaux

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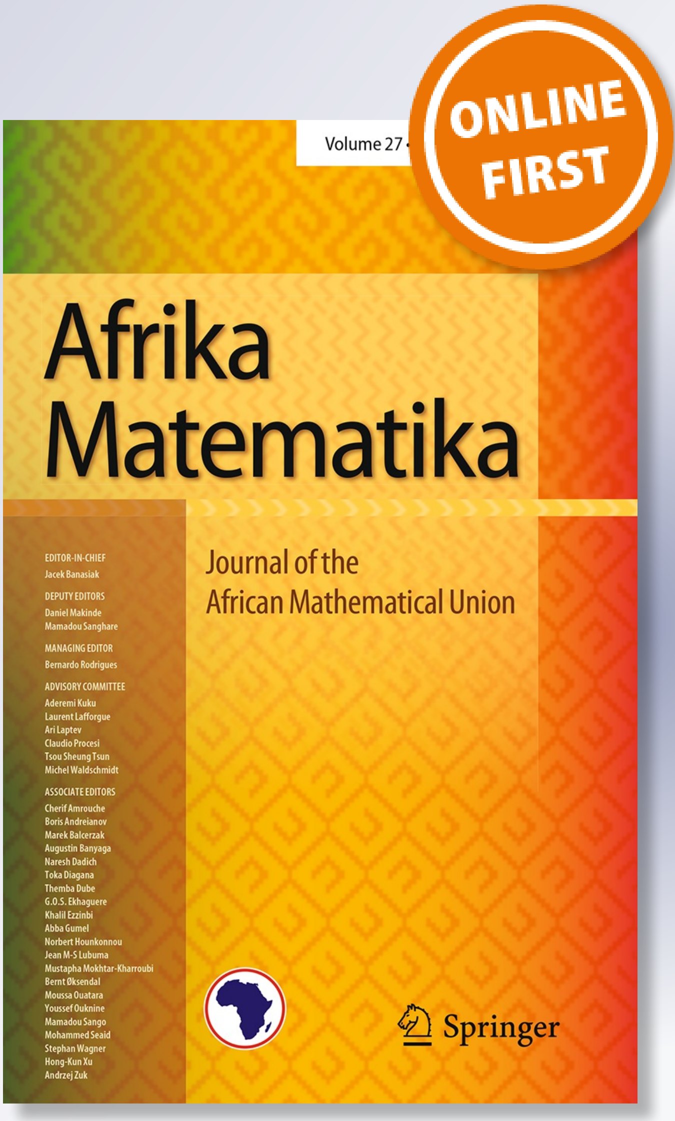
Afrika Matematika

ISSN 1012-9405

Afr. Mat.
DOI 10.1007/s13370-016-0393-4



Springer

Afr. Mat.
DOI 10.1007/s13370-016-0393-4

Les m-derivations des algebres de Lie de champs de vecteurs polynomiaux

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Received: 17 June 2014 / Accepted: 28 January 2016

Résumé Soit m un entier naturel supérieur ou égal à 2. On s'intéresse aux m-dérivations des algèbres de Lie $\mathfrak{P}$ de champs de vecteurs polynomiaux sur $\mathbb{R}^n$ qui contiennent tous les champs constants et le champ d'Euler. Si m est pair, elles sont des représentations adjointes par rapport au normalisateur de $\mathfrak{P}$. Pour m impair, toute m-dérivation est une somme de dérivée de Lie par rapport à un champ du normalisateur de $\mathfrak{P}$ et, de m-dérivation de $\mathfrak{P}$ qui à un champ quadratique fait corrrespondre un champ constant et qui s'annule sur les champs homogènes non quadratiques. On donne une condition nécessaire et suffisante pour qu'une application linéaire de $\mathfrak{P}$ vers elle-même soit de ce dernier type de m-dérivation. En général, sous une certaine condition sur $\mathfrak{P}$, toute m-dérivation du normalisateur de $\mathfrak{P}$ est intérieure. Ainsi, les m-dérivations de l'algèbre de Lie de tous les champs polynomiaux (resp. des champs affines) de $\mathbb{R}^n$ sont des m-dérivations intérieures.
Abstract Let be $m \geq 2$ a natural integer. We study the m-derivations of the Lie algebras $\mathfrak{P}$ of polynomial vector fields on $\mathbb{R}^n$ which contain all constant vector fields and the Euler's vector field. They are Lie derivative with respect to a $\mathbb{R}^n$ polynomial vector fields on the normalizer of $\mathfrak{P}$, when m is even. If m is odd, all m-derivation of $\mathfrak{P}$ is a sum of a Lie derivative with respect to a normalizer's vector fields and, a m-derivation of $\mathfrak{P}$ which acts on a quadratic vector fields to give a constant vector fields and which vanishes otherwise. We give a necessary and sufficient condition for a linear map of $\mathfrak{P}$ into it self to be a previous last type of m-derivation. Generally, under some conditions on $\mathfrak{P}$, all m-derivation of the normalizer of $\mathfrak{P}$ is inner. Hence, the m-derivation of Lie algebra of all polynomial vector fields respectively of affine vector fields on $\mathbb{R}^n$, is an inner m-derivation.

L'auteur bénéficie d'une bourse de recherches, initiative du Prof. Ingrid Daubechies en collaboration avec "Institute for the Conservation of Tropical Environments" (ICTE) Madagascar.

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Published online: 17 February 2016

Keywords m-dérivation · Algèbre de Lie graduée · Champs polynomiaux · Champ d'Euler · Champs affines

Mathematics Subject Classification 17B40 · 17B66 · 17B70

1 Introduction

Définition 1.1 Une $(m \geq 2)$-dérivation D d'une algèbre de Lie $\mathfrak{A}$ est une application $\mathbb{R}-$linéaire de $\mathfrak{A}$ dans $\mathfrak{A}$ telle que $\forall X_1, X_2, \ldots, X_m \in \mathfrak{A}$,

$$\begin{aligned} D\left[X_1,\left[X_2,\ldots,\left[X_{m-1},X_m\right]\ldots\right]\right] &= \left[D\left(X_1\right),\left[X_2,\ldots\left[X_{m-1},X_m\right]\ldots\right]\right] \\ &+ \left[X_1,\left[D\left(X_2\right),\ldots,\left[X_{m-1},X_m\right]\ldots\right]\right] + \ldots \\ &+ \left[X_1,\left[X_2,\ldots,\left[D\left(X_{m-1}\right),X_m\right]\ldots\right]\right] \\ &+ \left[X_1,\left[X_2,\ldots,\left[X_{m-1},D\left(X_m\right)\right]\ldots\right]\right]. \end{aligned} \tag{1.1}$$

On note $\mathrm{Der}^m(\mathfrak{A})$ l'algèbre de Lie des m-dérivations de $\mathfrak{A}$. Si $m = 2$, alors $\mathrm{Der}^2(\mathfrak{A})$ coïncide avec l'algèbre de Lie $\mathrm{Der}(\mathfrak{A})$ des dérivations de $\mathfrak{A}$. Dans le cas où $m = 3$, $\mathrm{Der}^3(\mathfrak{A})$ est l'ensemble des triple dérivations de $\mathfrak{A}$.

Définition 1.2 On dit qu'une m-dérivation de $\mathfrak{A}$ est intérieure par rapport à une algèbre de Lie $\mathfrak{B}$ contenant $\mathfrak{A}$, si elle est de la forme L_X, une dérivée de Lie par rapport à X élément de $\mathfrak{B}$. En particulier, L_X est une m-dérivation intérieure si $X \in \mathfrak{A}$. L'ensemble de ces m-dérivations est noté par $\mathrm{ad}_{\mathfrak{A}}$. Une m-dérivation D est dite sous-forme standard si elle est la somme d'une 2-dérivation de $\mathfrak{A}$ et d'une application $\mathbb{R}$-linéaire L de $\mathfrak{A}$ dans le centre de $\mathfrak{A}$ telle que $L[\mathfrak{A},[\mathfrak{A},[\ldots,[\mathfrak{A},\mathfrak{A}]\ldots]] = \{0\}$.

Les m-dérivations où $m \geq 2$ d'une algèbre de Lie sont naturellement une généralisation de la notion de dérivation et de triple dérivation. Récemment, [1] a traité les m-dérivations de l'algèbre des matrices triangulaires supérieures. Dans notre papier, on se propose d'étudier les m-dérivations d'une $\mathbb{R}$-algèbre de Lie $\mathfrak{P}$ formée de champs de vecteurs polynomiaux sur $\mathbb{R}^n$ contenant tous les champs constants et le champ d'Euler. D'après un résultat de [5], l'algèbre de Lie $\mathfrak{P}$ est graduée en une somme directe d'espaces vectoriels consistant à des champs polynomiaux homogènes de même degré supérieur ou égal à -1. On remarque que cette graduation est différente de celle de [3]. On sait que, toute 2-dérivation de $\mathfrak{P}$ est une dérivée de Lie par rapport à un élément du normalisateur $\mathfrak{N}$ de $\mathfrak{P}$. De plus, $\mathfrak{N}$ est une sous-algèbre de Lie de l'algèbre de Lie des champs de vecteurs polynomiaux de $\mathbb{R}^n$. Comme le centre de $\mathfrak{P}$ est réduit à zéro, alors toutes les m-dérivations sous-forme standard sont des dérivées de Lie par rapport à un champ de vecteurs de $\mathfrak{N}$. On généralise la manière d'étudier les 2-dérivations de $\mathfrak{P}$ dans [5], où m est inférieur ou égal au degré de nilpotence de $\mathfrak{P}$ (si $\mathfrak{P}$ n'est pas nilpotent alors ce degré est égal à $+\infty$). Car si m est dans un cas contraire, toute application $\mathbb{R}$-linéaire de $\mathfrak{P}$ est une m-dérivation de $\mathfrak{P}$. Ainsi, on constate que si $\mathfrak{P}$ contient des champs quadratiques, il existe des m-dérivations homogènes de degré -2 s'annulant sur les champs homogènes non quadratiques. Dans le cas où m est pair, ces derniers types de m-dérivations sont identiquement nulles. C'est-à-dire que l'algèbre de Lie des m-dérivations est identique à celle des 2-dérivations de $\mathfrak{P}$. Pour le cas où m est impair, on donne une condition nécessaire et suffisante pour qu'une application $\mathbb{R}$-linéaire de $\mathfrak{P}$, envoyant un champ quadratique à un champ constant et s'annulant autrement, soit une m-dérivation. Ce dernier type de m-dérivations est non-standard. Par ailleurs, un champ homogène de degré $k \geq -1$, $X = \alpha^{j}_{i_1<i_2<\cdots<i_p} x^{i_1}x^{i_2}\ldots x^{i_p}\frac{\partial}{\partial x^j}$ (ici on suit la convention d'Einstein sur la sommation d'indices) de $\mathfrak{P}$ est séparé dans $\mathfrak{P}$, si pour chaque i_0 et $j_1,\ j_2,\ldots,\ j_p$, $\alpha^{i_0}_{j_1<j_2<\cdots<j_p} x^{j_1}x^{j_2}\ldots x^{j_p}\frac{\partial}{\partial x^{i_0}} \in \mathfrak{P}$. Un sous-ensemble non vide de l'algèbre de Lie des champs polynomiaux est séparé si tous ses éléments homogènes non nuls sont séparés dans ce sous-ensemble. Par la présence du champ d'Euler dans $\mathfrak{P}$, tout champ quadratique de $\mathfrak{P}$ appartient à son idéal dérivé. En utilisant la propriété de la m-dérivation de degré -2 sur cet idéal, on détermine entièrement l'image d'un champ quadratique quelconque de $\mathfrak{P}$ par ce dernier type de m-dérivation, si $\mathfrak{P}$ est séparé.

Comme $\mathfrak{P}$ est inclus dans son normalisateur, en général, on peut alors déterminer toutes les m-dérivations de ce normalisateur. Cependant, on cherche des conditions pour n'avoir que des m-dérivations intérieures de $\mathfrak{N}$. En rectifiant une preuve de [5], on peut montrer que le normalisateur de $\mathfrak{N}$ coïncide avec $\mathfrak{N}$ lui-même si et seulement si $\mathfrak{P}$ est un idéal caractéristique de $\mathfrak{N}$. Dans ce cas, avec m pair, toute m-dérivation de $\mathfrak{N}$ est intérieure. En outre, on peut affirmer que si tous les champs diagonaux sont inclus dans $\mathfrak{P}$ (en particulier si l'idéal dérivé de $\mathfrak{P}$ coïncide avec $\mathfrak{P}$), alors toute m-dérivation (avec $m \geq 2$ un entier arbitraire) est intérieure par rapport à un champ de $\mathfrak{P}$. Si $\mathfrak{P}$ est séparé, tous les champs linéaires diagonaux appartiennent au normalisateur de $\mathfrak{P}$. Ainsi, toute m-dérivation de $\mathfrak{N}$ est une representation adjointe par rapport à $\mathfrak{N}$. Alors, on propose quelques illustrations des résultats par des exemples de m-dérivations non standards d'algèbres de Lie $\mathfrak{P}$, pour le cas où $\mathfrak{P}$ est séparé ou non séparé. Toutes les m-dérivations de l'algèbre de Lie de tous les champs polynomiaux (des champs affines) de $\mathbb{R}^n$ sont intérieures. De plus, on peut montrer que toute les m-dérivations d'une sous-algèbre de Lie de l'algèbre de Lie des champs affines contenant E et les champs constants sont des dérivations intérieures par rapport à un champ affine. Si les champs constants ou le champ d'Euler ne sont pas contenus dans une algèbre de Lie de champs de vecteurs polynomiaux, les résultats sur ses m-dérivations peuvent être différents. Un exemple d'une telle algèbre de Lie est celle des champs linéaires triangulaires supérieurs, qui est isomorphe à l'algèbre des matrices carrées triangulaires supérieures d'ordre n à coefficients réels cf. [4]. D'après un résultat de [1], la m-dérivation correspondante est sous-forme standard cf. Définition 1.2. Dans le même contexte, on peut aussi consulter d'autres exemples de dérivations dans la remarque 2.16 de [5] où le champ d'Euler n'appartient pas à $\mathfrak{P}$. Dans un autre article, on traitera les m-dérivations des algèbres de Lie attachées à une distribution involutive dans une variété différentiable.

2 Préliminaire

Soit $\mathfrak{P}$ une $\mathbb{R}$-sous-algèbre de Lie de l'algèbre de Lie des champs de vecteurs polynomiaux sur $\mathbb{R}^n$, contenant tous les champs constants et le champ d'Euler E. Dans la suite, on adopte la convention d'Einstein sur la sommation d'indices. En coordonnées locales $\left(x^i\right)_{1\leq i\leq n}$ de $\mathbb{R}^n$, le champ E s'écrit $x^i\frac{\partial}{\partial x^i}$. On note H_i l'ensemble des champs polynomiaux homogènes de degré $i \geq -1$. L'algèbre de Lie $\mathfrak{P}$ est graduée de la manière suivante:
$\mathfrak{P} = \oplus_{i\geq -1}\mathfrak{P}_i$, où tout $\mathfrak{P}_i = \mathfrak{P} \cap H_i$ est un sous-espace vectoriel de dimension finie de H_i, tels que

$$\left[\mathfrak{P}_{-1}, \mathfrak{P}_{-1}\right] = \{0\} \text{ et } \forall i,\ j \geq -1 \text{ , où } i + j \geq -1,\ \left[\mathfrak{P}_i, \mathfrak{P}_j\right] \subset \mathfrak{P}_{i+j}. \tag{2.1}$$

De plus, un champ X de $\mathfrak{P}_i$ est déterminé par $[E, X] = iX$ et tout $Y \in \mathfrak{P}$ s'écrit en $\sum_{-1\leq j\leq k} Y_j$ tel que $\forall j,\ Y_j \in \mathfrak{P}_j$ cf. [5]. Donc, on peut définir une m-dérivation D de $\mathfrak{P}$ à partir des images par D des champs homogènes de $\mathfrak{P}$.

Dans toute la suite, m est un entier naturel plus grand ou égal à 2. Il est clair que toute 2-dérivation d'une algèbre de Lie $\mathfrak{A}$ est une m-dérivation avec $m \geq 2$, en utilisant les définitions d'une m-dérivation. Alors, on a $\mathrm{Der}^2(\mathfrak{A}) \subset \mathrm{Der}^m(\mathfrak{A})$.

Nous désignons par $\chi(\mathbb{R}^n)$ l'algèbre de Lie des champs de vecteurs de $\mathbb{R}^n$ munie du crochet de champs de vecteurs. On rappelle que le normalisateur $\mathfrak{N}$ de $\mathfrak{P}$ est défini par $\mathfrak{N} = \{X \in \chi(\mathbb{R}^n) / [X, \mathfrak{P}] \subset \mathfrak{P}\}$, c'est une sous-algèbre des champs de vecteurs polynomiaux de $\mathbb{R}^n$. Comme $E \in \mathfrak{P}$, alors on a une somme directe d'espaces vectoriels $\mathfrak{N} = \mathfrak{P} \oplus \mathfrak{N}_0$, avec $\mathfrak{N}_0$ la partie d'éléments homogènes de degré 0 de $\mathfrak{N}$ non contenue dans $\mathfrak{P} - \{0\}$. En effet, $\mathfrak{P} \subset \mathfrak{N}$ est un idéal de $\mathfrak{N}$, alors le quotient $\mathfrak{N}/\mathfrak{P} \cong \mathfrak{N}_0$ est une algèbre de Lie et donc un espace vectoriel. Le centralisateur de $\mathfrak{P}$ qui est égal à $\{X \in \chi(\mathbb{R}^n) / [X, \mathfrak{P}] = \{0\}\}$, est réduit à zéro cf. [5].

Remarque 2.1 Dans le cas de l'algèbre de Lie $\mathfrak{P}$, une m-dérivation sous-forme standard est une 2-dérivation car le centralisateur de $\mathfrak{P}$ est nul. Alors, elle est sous une forme de dérivée de Lie par rapport à un champ du normalisateur de $\mathfrak{P}$ cf. [5].

Définition 2.2 Soit $\mathfrak{A}$ une algèbre de Lie, sa série centrale descendante est définie par $\mathfrak{C}^0(\mathfrak{A}) = \mathfrak{A}$ et pour tout $p > 0$, $\mathfrak{C}^p(\mathfrak{A}) = \left[\mathfrak{A}, \mathfrak{C}^{p-1}(\mathfrak{A})\right]$ cf. [2]. L'algèbre de Lie $\mathfrak{A}$ est nilpotente d'ordre $k > 0$ si k est le plus petit entier tel que $\mathfrak{C}^k(\mathfrak{A}) = \{0\}$. Cette série est décroissante au sens de l'inclusion: $\mathfrak{C}^{k+1}(\mathfrak{A}) \subset \mathfrak{C}^k(\mathfrak{A})$ pour tout $k \geq 0$.

Remarque 2.3 La série centrale descendante de $\mathfrak{P}$ vérifie $\mathfrak{C}^p(\mathfrak{P}) \neq \{0\}$ pour $0 \leq p \leq 1$. Cette assertion découle directement du fait que $\mathfrak{P}$ contient tous les champs constants et le champ d'Euler. En effet, $\mathfrak{C}^1(\mathfrak{P})$ contient l'ensemble de tous les champs constants.

Proposition 2.4 *Si l'algèbre de Lie $\mathfrak{P}$ est nilpotente d'ordre $k \geq 2$ avec $m > k$, alors toute application $\mathbb{R}$-linéaire de $\mathfrak{P}$ dans $\mathfrak{P}$ est une m-dérivation de $\mathfrak{P}$.*

Démonstration Compte tenu de la Remarque 2.3, $k \geq 2$, considérons la série centrale descendante de l'algèbre de Lie $\mathfrak{P}$. Alors si $m > k$, la relation (1.1) est triviale; d'où le résultat.

Alors, dans toute la suite, si $\mathfrak{P}$ est nilpotente d'ordre k, on ne s'intéresse qu'à ses m-dérivations avec $m \leq k$.

3 Etude des m-dérivations des algèbres de Lie $\mathfrak{P}$

Dans cette section, on examine les m-dérivations des algèbres de Lie attachées à $\mathfrak{P}$ et les applications des résultats obtenus.

Proposition 3.1 *Une m-dérivation D de $\mathfrak{P}$, nulle sur tous les champs constants et le champ d'Euler, est nulle sur tous les champs homogènes de degré 0 de $\mathfrak{P}$. Avec ces conditions, cette m-dérivation s'annule sur tous les champs quadratiques de $\mathfrak{P}$ si et seulement si D est nulle.*

Démonstration Supposons que $D(E) = D(C) = 0$, où $C \in H_{-1}$. On peut considérer $P \in \mathfrak{P}_k$ comme dans [5], où $k \in \mathbb{N} \cup \{-1\}$ et on raisonne par récurrence.

Pour $P \in H_{-1}$, on a par hypothèse $D(P) = 0$. De même, soit $P \in \mathfrak{P}_0$, on obtient

$$D[E, [E, [\ldots, [E, [C, P]] \ldots]]] = 0. \tag{3.1}$$

Par définition d'une m-dérivation (1.1) et par hypothèse pour tout C, on a $D(P) \in H_{-1}$. Or

$$D[E, [E, [\ldots, [E, [E, P]] \ldots]]] = 0,$$

par définition d'une m-dérivation et le résultat précédent, $D(P) = 0$.

Soient $k \geq 2$ et $P \in \mathfrak{P}_k$, supposons que pour tout $Q \in \mathfrak{P}_r$ où $-1 \leq r \leq k-1$, $D(Q) = 0$. Une relation analogue à (3.1) s'écrit

$$D\,[E, [E, [\ldots, [E, [C, P]]\ldots]]] = (k-1)^{m-2} D(Q),$$

où le champ Q appartient à $\mathfrak{P}_{k-1}$. Par hypothèse, $D(Q)$ est nul. Donc

$$[E, [\ldots, [E, [D(C), P]]\ldots]] + [E, [\ldots, [E, [C, D(P)]]\ldots]] = 0,$$

par la relation (1.1). On pose $D(P) = \sum_{i \geq -1} P_i'$ où chaque $P_i' \in \mathfrak{P}_i$. Alors pour tout $C \in H_{-1}$ et $i > -1$, on obtient la nullité de $(i-1)^{m-2}\left[C, P_i'\right]$. Ainsi, $\sum_{i \geq -1} P_i'$ est dans $\mathfrak{P}_{-1} \oplus \mathfrak{P}_1$. Or, on a l'égalité entre $D\,[E, [E, \ldots, [E, P]\ldots]]$ et $k^{m-1} D(P)$. Par conséquent et par définition, on a $k^{m-1}\left(P_{-1}' + P_1'\right) = (-1)^{m-1} P_{-1}' + P_1'$, ainsi $P_{-1}' = P_1' = 0$. D'où $\forall k \geq -1,\ D(P) = 0$ et $D \equiv 0$.

La réciproque est évidente. □

Proposition 3.2 *Soit D une m-dérivation de $\mathfrak{P}$ nulle sur E, vérifiant pour tout $i \neq 1, -1$ de $\mathbb{N} \cup \{-1\}$, $D(\mathfrak{P}_i) \subset \mathfrak{P}_i$, $D_{|\mathfrak{P}_{-1}}$ et $D_{|\mathfrak{P}_1}$ sont à valeurs dans $\mathfrak{P}_{-1} \oplus \mathfrak{P}_1$. Alors la m-dérivation s'écrit $D = \mathrm{L}_X + D_0$ avec X un champ de vecteurs polynomiaux homogène de degré 0, D_0 une m-dérivation nulle sur $\mathfrak{P}_i$ avec $i \neq -1, 1$ et que $D_0(\mathfrak{P}_{-1}) \subset \mathfrak{P}_1$, $D_0(\mathfrak{P}_1) \subset \mathfrak{P}_{-1}$.*

Démonstration Soit D une telle m-dérivation, alors pour $C \in \mathfrak{P}_{-1}$, $D(C) = W_{-1} + W_1$, où $W_{-1} \in \mathfrak{P}_{-1}$ et $W_1 \in \mathfrak{P}_1$. Comme D est une application $\mathbb{R}-$linéaire, alors il existe un seul $X \in H_0$ tel que $D(C) = [X, C] + W_1$ pour tout $C \in H_{-1}$. En effet, il suffit d'écrire l'équation $[X, C] = W_{-1}$ en coordonnées usuelles de $\mathbb{R}^n$ pour tout C et la résoudre. Par ailleurs, on obtient $D(E) = 0 = [X, E]$. Soient $V_1 \in \mathfrak{P}_1$, $V_0 \in \mathfrak{P}_0 - \{E\}$ et $C \in H_{-1}$, on a

$$D\,[V_0, [E, \ldots, [E, C]\ldots]] = [D(V_0), [E, \ldots, [E, C]\ldots]] + [V_0, [E, \ldots, [E, D(C)]\ldots]] \tag{3.2}$$

et

$$\begin{aligned} &D\,[C, [E, \ldots, [E, V_1]\ldots]] \\ &\quad = [[X, C] + W_1, [E, \ldots, [E, V_1]\ldots]] + [C, [E, \ldots, [E, D(V_1)]\ldots]]. \end{aligned} \tag{3.3}$$

Par la relation (3.2) et ce qui précède, on a

$$[X, [V_0, C]] = [D(V_0), C] + [V_0, [X, C]]. \tag{3.4}$$

En utilisant l'identité de Jacobi, on obtient $[X, V_0] - D(V_0) \in H_{-1} \cap H_0$. Ce qui entraîne la nullité de cette dernière expression pour tout $V_0 \in \mathfrak{P}_0$. Par conséquent, $D(V_0) = [X, V_0]$ pour tout $V_0 \in \mathfrak{P}_0$.

D'après ce qui précède, $D\,[C, [E, \ldots, [E, V_1]\ldots]] = [X, [C, V_1]]$. En identifiant l'égalité précédente avec la relation (3.3), $[W_1, V_1]$ et $[C, [X, V_1] - [E, \ldots, [E, D(V_1)]\ldots]]$ deviennent nuls, pour tout C de $\mathfrak{P}_{-1}$. Donc, on a $[X, V_1] - [E, \ldots, [E, D(V_1)]\ldots] \in H_{-1}$. En notant $D(V_1) = W_{-1}' + W_1'$, $D(V_1) = W_{-1}' + [X, V_1]$ pour tout $V_1 \in \mathfrak{P}_1$. Maintenant, soit $k \geq 2$ et on suppose que pour $-1 \leq r \leq k-1$, les $D(V_r) = [X, V_r] + W$ où $W \in \mathfrak{P}_{-1} \cup \mathfrak{P}_1$. C'est-à-dire, on raisonne par récurrence, prenons $V_k \in \mathfrak{P}_k$ et écrivons

$$\begin{aligned} &D\,[V_k, [E, \ldots, [E, C]\ldots]] \\ &\quad = [D(V_k), [E, \ldots, [E, C]\ldots]] + [V_k, [E, \ldots, [E, D(C)]\ldots]], \end{aligned}$$

qui donne en terme d'égalité entre les champs homogènes de même degré

$$D\,[V_k, C] = [D(V_k), C] + \left[V_k, W_{-1}\right]$$

Par hypothèse de récurrence et par identification, on déduit de cette égalité

$$[X, [V_k, C]] = [D(V_k), C] + [V_k, [X, C]] .$$

On utilise l'identité de Jacobi pour tout C et on a $D(V_k) - [X, V_k] \in \mathfrak{P}_k \cap \mathfrak{P}_{-1} = \{0\}$. Ainsi, D est la somme de L_X et d'une m-dérivation D_0 susmentionnée. □

Définition 3.3 On dit qu'une m-dérivation D de $\mathfrak{P}$ est homogène de degré k si $D(\mathfrak{P}_i) \subset \mathfrak{P}_{i+k}$ pour tout $i \geq -1$.

Proposition 3.4 *Une m-dérivation homogène de degré 0 sur $\mathfrak{P}$ qui s'annule en E est une dérivée de Lie par rapport à un champ polynomial homogène de degré 0.*

Démonstration La Proposition 3.2 est une extension de la présente proposition. En effet, il suffit de remarquer que dans ce cas, la m-dérivation D_0 est nulle. On peut aussi prouver notre proposition autrement en admettant la démonstration de la Proposition 3.2 où on se contente de chercher $D(V_k)$ avec $k = -1,\ 1$; alors la Proposition 3.1 permet de conclure. □

Remarque 3.5 On peut déduire de cette proposition qu'une dérivation homogène de degré 0 sur $\mathfrak{P}$ qui s'annule en E est une dérivée de Lie par rapport à un champ de vecteurs polynomial homogène de degré 0 cf. [5].

Théorème 3.6 *Toute m-dérivation D de $\mathfrak{P}$ où $m > 1$ un entier naturel pair, est intérieure par rapport au normalisateur $\mathfrak{N}$ de $\mathfrak{P}$. De plus, cette m-dérivation est égale à L_{F+X} avec $X \in H_0$ et $F \in \mathfrak{P} \ominus H_0$. Ainsi, on a des isomorphismes entre* $\mathrm{Der}^m(\mathfrak{P})$, $\mathrm{Der}(\mathfrak{P})$, $\mathrm{ad}_{\mathfrak{N}}$ *et* $\mathfrak{N}$.

Démonstration Soit D une m-dérivation de $\mathfrak{P}$. On considère $D(V_t)$, où V_t est un champ de $\mathfrak{P}$ homogène de degré $t \geq -1$.

On écrit comme dans [5], $D(V_t) = \sum_{-1\leq i\leq k} W_i$ et $D(E) = \sum_{-1\leq i\leq l} E_i$, avec $W_i,\ E_i \in \mathfrak{P}_i$.

Par définition d'une m-dérivation:

$$\begin{aligned} &D[E, [E, \ldots, [E, V_t] \ldots]] \\ &\quad = [D(E), [E, \ldots, [E, V_t] \ldots]] + \cdots + [E, [E, \ldots, [E, D(V_t)] \ldots]] . \end{aligned}$$

Alors, on obtient

- Pour $-1 \leq i \leq t-2$ et $i > t + l$, $\left(i^{m-1} - t^{m-1}\right) W_i = 0 \Rightarrow W_i = 0$ car m est pair.
- Pour $t - 1 \leq i \leq t + l$,

$$\left(i^{m-1} - t^{m-1}\right) W_i + \left(t^{m-2} + t^{m-3} i + t^{m-4} i^2 + \ldots + i^{m-2}\right) \left[E_{i-t}, V_t\right] = 0 \tag{3.5}$$

et

$$[E_0, V_t] = 0. \tag{3.6}$$

Par factorisation et simplification, on a:

$$\begin{aligned} D(V_t) &= W_t + \sum_{t-1\leq i\leq t+l} \left(\frac{1}{t-i}\right) \left[E_{i-t}, V_t\right] \\ &= W_t + \left[\sum_{i\neq 0} \left(\frac{-1}{i}\right) E_i, V_t\right]. \end{aligned} \tag{3.7}$$

En notant $F = \sum_{-1\leq i\leq l, i\neq 0} \left(\frac{-1}{i}\right) E_i$, qui est un élément de $\mathfrak{P}$, on obtient $D(V_t) = W_t + [F, V_t]$.

On pose $D' = D - \mathrm{L}_F$, c'est bien une m-dérivation homogène de degré 0 de $\mathfrak{P}$. D'après (3.6) et le fait que $\mathfrak{P}$ contient tous les champs constants, on a $[E_0, C] = 0 \; \forall C \in H_{-1}$. Par conséquent, $E_0 = 0$ et $D'(E) = 0$. D'après la Proposition 3.4, $D' = \mathrm{L}_X$ où $X \in H_0$. Alors, la m-dérivation D est une dérivée de Lie par rapport à un champ de vecteurs polynomiaux de $\mathbb{R}^n$, qui est $F + X$. Par conséquent, $F + X$ appartient au normalisateur $\mathfrak{N}$ de $\mathfrak{P}$, avec F élément de $\mathfrak{P} - H_0$. Ainsi, pour tout m pair, $\mathrm{Der}^m(\mathfrak{P}) = \mathrm{ad}_{\mathfrak{N}}$. Ces dernières algèbres de Lie sont isomorphes à $\mathfrak{N}$ et $\mathrm{Der}(\mathfrak{P})$ en adaptant une preuve de [5]. $\square$

Remarque 3.7 Dans le cas restrictif $m = 2$, ce théorème est exactement celui de [5] pour les dérivations de $\mathfrak{P}$.

Dans toute la suite, on suppose que $m \geq 3$ impair.

Proposition 3.8 *Soit D une m-dérivation nulle sur $\mathfrak{P}_i$ avec $i \neq -1, 1$ et que $D(\mathfrak{P}_{-1}) \subset \mathfrak{P}_1$, $D(\mathfrak{P}_1) \subset \mathfrak{P}_{-1}$. Etant donnés $(V_{-1}, V_1) \in \mathfrak{P}_{-1} \times \mathfrak{P}_1$:*

1. *On a $D(V_{-1}) = 0$.*
2. *Pour tout $V_t \in \mathfrak{P}_{t \neq 0}$, $[D(V_1), V_t] = 0$.*
3. *Pour tout $V_0 \in \mathfrak{P}_0$, on a $[V_0, D(V_1)] = -D[V_0, V_1]$.*

Démonstration Soient $t, l \in \mathbb{N} \cup \{-1\}$ et $(V_l, V_t') \in \mathfrak{P}_l \times \mathfrak{P}_t$, et D une telle m-dérivation. On écrit

$$\begin{aligned} D\left[V_t', [E, [\ldots, [E, V_l] \ldots]\right] = &\left[D\left(V_t'\right), [E, [\ldots, [E, V_l] \ldots]\right] \\ &+ \left[V_t', [E, [\ldots, [E, D(V_l)] \ldots]\right]. \end{aligned} \tag{3.8}$$

On identifie (3.8) en terme de champs homogènes de même degré et on suppose que:

1. $l = 1$ et $t \geq 2$, alors on a $\left[D(V_1), V_t'\right] = 0$.
2. $l = 1$ et $t = 0$, on a $\left[V_0', D(V_1)\right] = -D\left[V_0', V_1\right]$.
3. $l = 1$ et $t = 1$, on doit avoir $\left[D(V_1), V_1'\right] = \left[V_1, D\left(V_1'\right)\right]$.
4. $l = -1$ et $t = -1$, alors $\left[D(V_{-1}), V_{-1}'\right] = \left[V_{-1}, D\left(V_{-1}'\right)\right]$.

Or, pour $V_{-1}'' \in \mathfrak{P}_{-1}$,

$$D\left[V_{-1}'', \left[V_{-1}', \left[E, \ldots, [E, V_{-1}] \ldots\right]\right]\right] = \left[V_{-1}'', \left[D\left(V_{-1}'\right), V_{-1}\right] + \left[V_{-1}', D(V_{-1})\right]\right]. \tag{3.9}$$

D'après la relation 4. et pour tout V_{-1}'' de $\mathfrak{P}_{-1}$, (3.9) donne $\left[V_{-1}', D(V_{-1})\right] = 0$ et $D(V_{-1}) = 0$.

Par ailleurs,

$$\begin{aligned} D\left[V_{-1}, \left[V_1', [E, [\ldots, [E, V_1] \ldots]]\right]\right] = &\left[D(V_{-1}), \left[V_1', [E, [\ldots, [E, V_1] \ldots]]\right]\right] \\ &+ \left[V_{-1}, \left[D\left(V_1'\right), [E, [\ldots, [E, V_1] \ldots]]\right]\right] \\ &+ \left[V_{-1}, \left[V_1', [E, [\ldots, [E, D(V_1)] \ldots]]\right]\right] \end{aligned}$$

$\forall \left(V_{-1}, V_1', V_1\right) \in \mathfrak{P}_{-1} \times \mathfrak{P}_1 \times \mathfrak{P}_1$. On peut dire que $D\left[V_{-1}, \left[E, \left[E, \left[\ldots, \left[E, [V_1', V_1]\right] \ldots\right]\right]\right]\right] = 0$ où E se repète $m - 2$ fois. Alors, on obtient $\left[V_{-1}, \left[V_1', D(V_1)\right]\right] = 0$ d'après 3. de la présente preuve. Ainsi, on établit l'égalité $\left[V_1', D(V_1)\right] = 0$. D'où les résultats. $\square$

Théorème 3.9 *Pour $m \geq 3$ impair, toute m-dérivation D de $\mathfrak{P}$ est la somme d'une dérivée de Lie par rapport à un champ du normalisateur de $\mathfrak{P}$ et d'une m-dérivation homogène de degré -2 nulle sur $\mathfrak{P}_{i \neq 1}$.*

Démonstration On peut faire le même raisonnement que dans la preuve du Théorème 3.6 avec les modifications suivantes:

- Pour $-1 \leq i \leq t-2$ et $i > t+l$, on a $\left(i^{m-1} - t^{m-1}\right) W_i = 0$: dans le cas où $t = 1$ et $i = -1$, on a une indétermination sur W_i, sinon $W_i = 0$.
- Pour $t-1 \leq i \leq t+l$, on obtient (3.5), et on a une indétermination sur W_i si $t = -1$ et $i = 1$, sinon W_i est bien déterminé. On a $\left[E_2, V_{-1}\right] = 0$ pour tout V_{-1}; ce qui entraine la nullité de E_2.

On obtient (3.7), pour tout $t \neq -1,\ 1$.

Notant à nouveau $F = \sum_{-1 \leq i \leq l, i \neq 0} \left(\frac{-1}{i}\right) E_i \in \mathfrak{P}$, on a $D(V_t) = W_t + [F, V_t]$ pour tout $t \neq -1,\ 1$.

On pose $D' = D - \mathrm{L}_F$, c'est une m-dérivation vérifiant les hypothèses de la Proposition 3.2. En effet, d'après (3.6) et le fait que tous les champs constants sont dans $\mathfrak{P}$, on a $[E_0, C] = 0\ \forall C \in H_{-1}$. En conséquence, $E_0 = 0$ et $D'(E) = 0$. D'après la Proposition 3.2, $D' = \mathrm{L}_X + D_0$ où $X \in H_0$ et D_0 une m-dérivation définie par la Proposition 3.2. Cette application D_0 est nulle sur $\mathfrak{P}_{-1}$ d'après la Proposition 3.8. Alors, $D = \mathrm{L}_{F+X} + D_0$ avec $F + X$ est un champ de vecteurs polynomial de $\mathbb{R}^n$. Par le même raisonnement que celui de la preuve du Théorème 3.6, on montre que $F + X \in \mathfrak{N}$ où F appartient à $\mathfrak{P} \ominus \mathfrak{P}_0$. □

Théorème 3.10 *On suppose que $m \geq 3$ est impair. Pour qu'une application $\mathbb{R}$-linéaire D de $\mathfrak{P}$, nulle sur $\mathfrak{P} \ominus \mathfrak{P}_1$ et dont l'image est incluse dans $\mathfrak{P}_{-1}$, soit une m-dérivation, il faut et il suffit que:*

$$[D(\mathfrak{P}_1), \mathfrak{P} - \mathfrak{P}_0] = \{0\}; \tag{3.10}$$

$$D[X, Y] = [D(Y), X]\ \forall (X, Y) \in \mathfrak{P}_0 \times \mathfrak{P}_1; \tag{3.11}$$

$$[D(\mathfrak{P}_1), [\mathfrak{P}, \mathfrak{P}] \cap \mathfrak{P}_0] = \{0\}; \tag{3.12}$$

si $\left[X_1, \left[X_2, \ldots, \left[X_i, \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right]\right] \in \mathfrak{P}_1$ où i le premier indice tel que $X_i \in \mathfrak{P}_1$ avec l'existence de $1 \leq j < i$ tels que $X_j \in \mathfrak{P}_{-1} \cup \mathfrak{P}_{t \geq 2}$, alors

$$D\left[X_1, \left[X_2, \ldots, \left[X_i, \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right]\right] = 0 \tag{3.13}$$

Démonstration On suppose que D est une m-dérivation de $\mathfrak{P}$ nulle sur $\mathfrak{P} \ominus \mathfrak{P}_1$ et telle que $D(\mathfrak{P}_1) \subset \mathfrak{P}_{-1}$. Les deux premières conditions nécessaires sont obtenues en utilisant la Proposition 3.8. En outre pour $(X,\ X_1,\ X_2)$ dans $\mathfrak{P}_1 \times \mathfrak{P} \times \mathfrak{P}$ avec $[X_1, X_2] \in \mathfrak{P}_0$, on applique D à $[E, \ldots, [E, [X, [X_1, X_2]] \ldots]]$. On a alors $[D(X), [X_1, X_2] \ldots] = 0$ par la relation 3. de la Proposition 3.8. Pour montrer (3.13), soient $X_1, X_2, \ldots, X_j, \ldots, X_i, \ldots, X_m$ vérifiant les conditions susmentionnées. Si $\left[X_{i+1}, \ldots, \left[X_{m-1}, X_m\right] \ldots\right] \in \mathfrak{P}_0$, $[D(X_i), \ldots, \left[X_{m-1}, X_m\right] \ldots]$ est égal à $D(Y \in \mathfrak{P}_1)$ par (3.11), sinon est égal à 0 par (3.10). Par les relations (3.11) et (3.10), $\left[X_{j+1}, \left[X_{j+2}, \ldots, \left[D(X_i), \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right]\right] = D(Z \in \mathfrak{P}_1)$ ou 0 (cette technique (T) sera utilisée dans la suite de notre démonstration). Or $X_j \in \mathfrak{P}_{-1} \cup \mathfrak{P}_{t \geq 2}$, alors

$$\left[X_j, \left[X_{j+1}, \ldots, \left[D(X_i), \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right]\right] = 0$$

par (3.10). A l'aide de (T), on peut affirmer aussi que

$$\left[X_1, \left[X_j, \ldots, \left[D(X_k), \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right]\right] = 0,\ \forall k \geq i \text{ où} X_k \in \mathfrak{P}_1.$$

Ainsi, la relation (3.13) est obtenue en utilisant (1.1).

Par construction, un endomorphisme D de $\mathfrak{P}$ qui s'annule sur $\mathfrak{P} \ominus \mathfrak{P}_1$ et tel que $D(\mathfrak{P}_1) \subset \mathfrak{P}_{-1}$; est une m-dérivation si et seulement si les trois assertions suivantes sont satisfaites:

(a) Pour $X_1, \ldots, X_m \in \mathfrak{P} - \mathfrak{P}_1$ tels que $\left[X_1, \left[X_2, \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right] = V_1 \in \mathfrak{P}_1$, alors $D\left(V_1\right)$ est nul.

(b) Pour $X_{i_1}, \ldots, X_{i_p} \in \mathfrak{P}_1$ avec $i_1 < i_2 < \cdots < i_p$ et les autres $X_{i_{p+1}}, \ldots, X_{i_m} \in \mathfrak{P} - \mathfrak{P}_1$, où les indices deux à deux différents i_k sont dans $\{1, \ldots, m\}$ tels que

$$\left[X_1, \left[X_2, \ldots, \left[X_{m-1}, \ldots, X_m\right] \ldots\right]\right] \notin \mathfrak{P}_1,$$

alors

$$\begin{aligned} &\left[X_1, \left[X_2, \ldots, \left[D\left(X_{i_1}\right), \ldots, \left[X_{i_k} \ldots, \left[X_{m-1}, \ldots, X_m\right] \ldots\right]\right]\right]\right] + \ldots \\ &\quad + \left[X_1, \left[X_2, \ldots, \left[X_{i_1}, \ldots, \left[X_{i_k} \ldots, \left[D\left(X_{i_p}\right), \ldots, X_m\right] \ldots\right]\right]\right]\right] = 0. \end{aligned} \tag{3.14}$$

(c) Pour $X_{i_1}, \ldots, X_{i_p} \in \mathfrak{P}_1$ avec $i_1 < i_2 < \ldots < i_p$ et les autres $X_{i_{p+1}}, \ldots, X_{i_m} \in \mathfrak{P} - \mathfrak{P}_1$, où les indices deux à deux différents i_k sont dans $\{1, \ldots, m\}$ tels que

$$\left[X_1, \left[X_2, \ldots, \left[X_{m-1}, \ldots, X_m\right] \ldots\right]\right] = V_1 \in \mathfrak{P}_1,$$

alors

$$\begin{aligned} D\left(V_1\right) = &\left[X_1, \left[X_2, \ldots, \left[D\left(X_{i_1}\right), \ldots, \left[X_{i_k} \ldots, \left[X_{m-1}, \ldots, X_m\right] \ldots\right]\right]\right]\right] + \ldots \\ &+ \left[X_1, \left[X_2, \ldots, \left[X_{i_1}, \ldots, \left[X_{i_k} \ldots, \left[D\left(X_{i_p}\right), \ldots, X_m\right] \ldots\right]\right]\right]\right]. \end{aligned} \tag{3.15}$$

En effet, il suffit de vérifier la relation (1.1) pour D à partir de a) b) c) et le cas trivial qu'on appelle d) où $X_1, \ldots, X_m \in \mathfrak{P} - \mathfrak{P}_1$ tels que $\left[X_1, \left[X_2, \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right] \notin \mathfrak{P}_1$.

Maintenant, démontrons la réciproque de notre théorème à partir de ces dernières considérations. On suppose que (3.10), (3.11), (3.12) et (3.13) soient satisfaites et montrons que a), b), c) et d) sont vraies pour l'endomorphisme D.

Les preuves de a) et d) pour D sont immédiates. Les preuves de b) et de c) qui vont suivre sont divisées en 3 astérisques résumant tous les cas possibles correspondants à b) et 3 astérisques résumant tous ceux de c).

– Pour la démonstration de b), considérons $\left[X_1, \left[X_2, \ldots, \left[X_{m-1}, \ldots, X_m\right] \ldots\right]\right] \notin \mathfrak{P}_1$ où il existe un premier indice i_1 tel que X_{i_1} appartient à $\mathfrak{P}_1$:

  $*$ Si $i_1 = 1$, alors $\left[X_2, \ldots, \left[X_{m-1}, \ldots, X_m\right] \ldots\right] \notin \mathfrak{P}_0$. En vertus de (3.10),

$$\left[D\left(X_1\right), \left[X_2, \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right] = 0.$$

  Par (T) obtenu à partir de (3.11) et (3.10), on aura

$$\left[X_1, \left[X_2, \ldots, \ldots, \left[D\left(X_{i_k}\right) \ldots, \left[X_{i_p}, \ldots, X_m\right] \ldots\right]\right]\right] = 0 \; \forall 1 < k \leq p.$$

  On obtient alors (3.14) en sommant ces expressions.

  $*$ Si les $X_1, \ldots, X_{i_1-1}$ $(i_1 > 1)$ appartiennent à $\mathfrak{P}_0$, alors $\left[X_{i_1+1}, \ldots, \left[X_{m-1}, \ldots, X_m\right] \ldots\right] \notin \mathfrak{P}_0$. Par conséquent, $\left[X_1, \ldots, \left[D\left(X_{i_1}\right), \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right] = 0$ par (3.10). De même, pour tout $1 < k \leq p$, $\left[X_1, \ldots, \left[D\left(X_{i_k}\right), \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right] = 0$ d'après (T). On en tire l'égalité (3.14).

  $*$ Si j est le dernier indice tel que $X_{j<i_1} \in \mathfrak{P}_{-1} \cup \mathfrak{P}_{t \geq 2}$, on a en utilisant (T),

$$\text{pour } 1 \leq k \leq p, \; \left[X_1, \ldots, \left[X_j, \ldots, \left[D\left(X_{i_k}\right), \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right]\right] = 0.$$

  Ainsi, on peut affirmer que (3.14) est vrai.

– Pour la preuve de c), soient $\left[X_1, \left[X_2, \ldots, \left[X_{m-1}, \ldots, X_m\right] \ldots\right]\right] \in \mathfrak{P}_1$ où X_{i_1} est le premier champ qui appartient à $\mathfrak{P}_1$:

$*$ Si tous les $X_1, \ldots, X_{i_1-1} \in \mathfrak{P}_0$ avec $i_1 \neq m$ alors $\left[X_{i_1+1}, \ldots, \left[X_{m-1}, X_m\right] \ldots\right] \in \mathfrak{P}_0$. D'après (3.11),

$$\begin{aligned} &D\left[X_1, \ldots, \left[X_{i_1}, \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right] \\ &\quad = (-1)^{i_1}\left[X_1, \ldots, \left[D\left(X_{i_1}\right), \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right]. \end{aligned}$$

On aura à l'aide de (T)

$$\text{pour } 1 < k \leq p,\ \left[X_1, \ldots, \left[D\left(X_{i_k}\right), \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right] = 0.$$

En rassemblant ces deux resultats, on obtient (3.15) pour i_1 pair. Pour i_1 impair, d'après (3.12)

$$\left[X_1, \ldots, \left[D\left(X_{i_1}\right), \ldots, \left[X_{m-1}, X_m\right] \ldots\right]\right] = 0,$$

et on a encore (3.15).

$*$ Si les $X_1, \ldots, X_{i_1-1} \in \mathfrak{P}_0$ avec $i_1 = m$. En vertus de (T), on obtient

$$D\left[X_1, \ldots, \left[X_{m-1}, X_m\right] \ldots\right] = (-1)^{m-1}\left[X_1, \ldots, \left[X_{m-1}, D\left(X_m\right)\right] \ldots\right].$$

Ainsi, l'égalité (3.15) est vraie.

$*$ Supposons que j soit le dernier indice tel que $X_{j<i_1} \in \mathfrak{P}_{-1} \cup \mathfrak{P}_{t\geq 2}$. Alors, le premier membre de l'égalité (3.15) est nul par (3.13). De même, chaque élément de la somme dans son deuxième membre est identiquement nul d'après (T). Ainsi, la relation (3.15) est satisfaite.

D'où, D est une m-dérivation de $\mathfrak{P}$ et on achève la preuve. □

Dans toute la suite, on notera par $\langle Z\rangle$ l'$\mathbb{R}$-espace vectoriel engendré par un élément $Z \in \mathfrak{P}$.

Théorème 3.11 *Soit D une m-dérivation de $\mathfrak{P}$, de degré -2 et nulle sur $\mathfrak{P}_i$ où $i \neq 1$. Pour $V_1 \in \mathfrak{P}_1 - \{0\}$:*

- *On suppose que les éléments de $\mathfrak{P}_1$ sont séparés. S'il existe $X \in \mathfrak{P}_0 \ominus \langle E\rangle$ tel que $[X, V_1] = V_1$, alors $D(V_1) = 0$.*
- *Les éléments de $\mathfrak{P}_1$ sont supposés séparés. S'il existe $\left(X, V_1'\right) \in (\mathfrak{P}_0 \ominus \langle E\rangle) \times \mathfrak{P}_1$ tel que $\left[X, V_1'\right]$ est égal à $V_1 \neq V_1'$, où $\operatorname{div}\left(V_1'\right) \neq 0$ ou la composante de V_1' est différente de celle de V_1 à multiplication par une constante près, alors $D(V_1)$ est nul. Dans le cas contraire, on peut choisir un $D(V_1)$ non nul sauf s'il y a des hypothèses supplémentaires.*
- *Si pour chaque $X \in \mathfrak{P}_0 \ominus \langle E\rangle$, $[X, V_1] = 0$, alors $[D(V_1), Y] = 0$ pour tout $Y \in \mathfrak{P} \ominus \langle E\rangle$.*
- *S'il existe $X, Y \in \mathfrak{P} - \mathfrak{P}_0$ tels que $[X, Y] = V_1$, alors $D(V_1) = 0$.*

Démonstration Soient $(x^i)_i$ un système de coordonnées de $\mathbb{R}^n$ et $V_1 \in \mathfrak{P}_1 - \{0\}$. Pour simplifier, on montre les résultats de chaque tiret pour un champ fixé.

Pour la démonstration du premier tiret, on suppose que l'ensemble $\mathfrak{P}_1$ est séparé. S'il existe $X \in \mathfrak{P}_0 \ominus \langle E\rangle$ tel que $[X, V_1] = V_1$, alors V_1 est de l'un des types de champs quadratiques suivants à une multiplication par une constante près:

- Pour $V_1 = \left(x^1\right)^2 \frac{\partial}{\partial x^1}$, l'existence de tous les champs constants dans $\mathfrak{P}$ entraîne qu'on peut trouver un $X = x^1 \frac{\partial}{\partial x^1} \in \mathfrak{P}$ tel que $[X, V_1] = V_1$. D'après la relation (3.11), $D(V_1) = -[X, D(V_1)] = \alpha \frac{\partial}{\partial x^1}$. En vertus de l'égalité (3.10), on obtient $\alpha = 0$.

- Pour $V_1 = x^1x^2\frac{\partial}{\partial x^1}$, s'il existe $X \in \mathfrak{P}_0 \ominus \langle E\rangle$ tel que $[X, V_1] = V_1$, alors $X = x^2\frac{\partial}{\partial x^2}$. D'après l'égalité (3.11), $D(V_1) = -\left[x^2\frac{\partial}{\partial x^2}, D(V_1)\right] = \alpha\frac{\partial}{\partial x^1}$. Par la relation (3.10), on aura $\alpha = 0$.
- Pour $V_1 = x^1x^2\frac{\partial}{\partial x^t}$ avec $t \neq 1,\ 2$, s'il existe $X \in \mathfrak{P}_0 \ominus \langle E\rangle$ tel que $[X, V_1] = V_1$, alors $X = x^1\frac{\partial}{\partial x^1}$ ou $X = x^2\frac{\partial}{\partial x^2}$ ou $X = -x^t\frac{\partial}{\partial x^t}$. Dans ces cas, les relations (3.11) et (3.10) nous donnent $D(V_1) = 0$.
- Pour $V_1 = \left(x^1\right)^2\frac{\partial}{\partial x^2}$, s'il existe $X \in \mathfrak{P}_0 \ominus \langle E\rangle$ tel que $[X, V_1] = V_1$, alors $X = \frac{1}{2}x^1\frac{\partial}{\partial x^1}$ ou $X = -x^2\frac{\partial}{\partial x^2}$. Par les mêmes raisons que précédemment, on peut obtenir $D(V_1) = 0$.

  Pour la preuve du deuxième tiret, s'il existe $X \in \mathfrak{P}_0 \ominus \langle E\rangle$ tel que $\left[X, V_1'\right] = V_1 \neq V_1'$, on procède comme suit:
- Pour $V_1' = \left(x^1\right)^2\frac{\partial}{\partial x^1}$, les V_1 non nuls possibles sont de la forme $\alpha x^1x^t\frac{\partial}{\partial x^1}$ ou $\beta\left(x^1\right)^2\frac{\partial}{\partial x^t}$ avec $t \neq 1$ et α, β des réels. Ainsi, $D(V_1) = 0$ en utilisant le résultat précédent et la relation (3.11).
- Pour $V_1' = x^1x^2\frac{\partial}{\partial x^1}$, les champs V_1 non nuls possibles sont de la forme $\alpha\left(x^2\right)^2\frac{\partial}{\partial x^1}$ ou $\beta\left(\left(x^1\right)^2\frac{\partial}{\partial x^1} - x^1x^2\frac{\partial}{\partial x^2}\right)$, $\alpha_0x^tx^2\frac{\partial}{\partial x^1}$, $\beta_0x^1x^t\frac{\partial}{\partial x^1}$ avec $t \neq 1,\ 2$ et $\alpha, \alpha_0, \beta, \beta_0$ des réels. Ainsi, $D(V_1) = 0$ en vertus des résultats précédents et de l'égalité (3.11) avec des champs X de $\mathfrak{P}$ convenables.
- Pour $V_1' = x^1x^2\frac{\partial}{\partial x^t}$ avec $t \neq 1,\ 2$, les champs V_1 non nuls possibles sont de la forme $\alpha\left(x^1\right)^2\frac{\partial}{\partial x^t}$, $\alpha\left(x^2\right)^2\frac{\partial}{\partial x^t}$, $\beta\left(x^tx^2\frac{\partial}{\partial x^t} - x^1x^2\frac{\partial}{\partial x^1}\right)$, $\beta_0\left(x^tx^1\frac{\partial}{\partial x^t} - x^1x^2\frac{\partial}{\partial x^2}\right)$, $\alpha_0x^ix^2\frac{\partial}{\partial x^t}$, $\beta_1x^ix^1\frac{\partial}{\partial x^t}$, $\alpha_1x^1x^2\frac{\partial}{\partial x^i}$, où $\alpha, \alpha_0,\ \alpha_1,\ \beta,\ \beta_0,\ \beta_1$ des réels et $i \neq 1,\ 2,\ t$. En utilisant les résultats précédents, la relation (3.11) et en choisissant des champs X de $\mathfrak{P}$ convenables, $D(V_1) = 0$ sauf pour le dernier champ où on a $D(V_1) = \alpha_1\alpha_2\frac{\partial}{\partial x^i}$ ($\alpha_2 \in \mathbb{R}$). Si on peut trouver un champ non nul de $\mathfrak{P}_0$ ayant une composante comportant x^i, alors ce dernier $D(V_1) = 0$ d'après la relation (3.11).
- Pour $V_1' = \left(x^1\right)^2\frac{\partial}{\partial x^t}$, les V_1 possibles non nuls sont de la forme $\alpha\left(2x^1x^t\frac{\partial}{\partial x^t} - \left(x^1\right)^2\frac{\partial}{\partial x^1}\right)$ ou $\beta x^1x^i\frac{\partial}{\partial x^t}$, $\alpha_0\left(x^1\right)^2\frac{\partial}{\partial x^i}$ avec $t \neq 1$, $i \neq t,\ 1$ avec $\alpha, \beta,\ \alpha_0$ des réels. Ainsi, $D(V_1) = 0$ d'après le résultat précédent et la relation (3.11), sauf pour le dernier champ où $D(V_1) = \alpha_1\alpha_0\frac{\partial}{\partial x^i}$ ($\alpha_1 \in \mathbb{R}$). Mais s'il existe un champ non nul de $\mathfrak{P}_0$ ayant une composante x^i, alors par l'égalité venant de (3.11), $D(V_1)$ devient nul.

  Pour démontrer le troisième tiret, on exploite les relations (3.10), (3.11). Le dernier tiret se montre en utilisant l'égalité (3.13) du Théorème 3.10. □

On identifie d'une manière unique un champ de H_0 à un élément de $gl(n, \mathbb{R})$ cf. [4] p.5. On l'appelera champ linéaire. On suppose dans la suite, que m est un entier naturel supérieur ou égal à 2.

Corollaire 3.12 *On a les résultats suivants:*

- *Le normalisateur de $\mathfrak{N}$ coïncide avec $\mathfrak{N}$ lui-même si et seulement si $\mathfrak{P}$ est un idéal caractéristique de $\mathfrak{N}$, dans ce cas, toute m-dérivation de $\mathfrak{N}$ est intérieure pour m pair.*
- *Si l'ensemble des champs linéaires diagonaux H_0^d est inclus dans $\mathfrak{P}_0$, alors toute m-dérivation de $\mathfrak{P}$ est intérieure. En particulier, on aura les mêmes résultats si l'idéal dérivé $[\mathfrak{P}, \mathfrak{P}]$ coïncide avec $\mathfrak{P}$.*
- *Si $\mathfrak{P}$ est séparé, alors toute m-dérivation du normalisateur de $\mathfrak{P}$ est intérieure dans son normalisateur.*

Démonstration Soient D une dérivation de $\mathfrak{N}$ et $\mathcal{N}$ le normalisateur de $\mathfrak{N}$. D'après nos théorèmes, $D = \mathrm{L}_X$ avec $X \in \mathcal{N}$. Ainsi, $\mathfrak{P}$ est un idéal caractéristique de $\mathfrak{N}$ équivaut à $\mathfrak{P}$ est un idéal de $\mathcal{N}$. Par définition d'un normalisateur, cette assertion est équivalente à $\mathcal{N} = \mathfrak{N}$. Le reste de la preuve de ce premier tiret découle de ce dernier résultat et de nos théorèmes.

Soit D une m-dérivation de $\mathfrak{P}$. Par le fait que $H_0^d \subset \mathfrak{P}_0$, $\mathfrak{P}$ est séparé. Pour tout $V_1 \in \mathfrak{P}_1$, il existe $X \in \mathfrak{P}_0 \ominus \langle E \rangle$ tel que $[X, V_1] = V_1$. D'après le Théorème 3.11, le Théorème 3.9 et le Théorème 3.6, D est une dérivée de Lie par rapport à un champ du normalisateur de $\mathfrak{P}$. On peut même dire que cette m-dérivation est intérieure par rapport à $\mathfrak{P}$ d'après la fin de la preuve du Théorème 2.12 de [5]. Dans ce cas, le normalisateur de $\mathfrak{P}$ est $\mathfrak{P}$ lui-même. Par ailleurs, si $[\mathfrak{P}, \mathfrak{P}] = \mathfrak{P}$, alors tous les champs diagonaux sont dans $\mathfrak{P}$ et on a le résultat d'après ce qu'on a démontré précédemment.

Si $\mathfrak{P}$ est séparé, alors H_0^d est inclus dans le normalisateur de $\mathfrak{P}$. Or, le normalisateur de $\mathfrak{P}$ est une algèbre de Lie de champs polynomiaux de $\mathbb{R}^n$, alors, on a le résultat d'après une partie de la démonstration de l'assertion du tiret précédent de cette preuve. □

Remarque 3.13 La démonstration du Corollaire 2.13 de [5] est rectifiée par celle du premier tiret du corollaire précédent. En effet, les champs diagonaux ne sont pas forcémment dans le normalisateur de $\mathfrak{P}$; comme le montre l'exemple suivant. Sur $\mathbb{R}^4$, soit $\mathfrak{P}$ l'algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}, E, x\frac{\partial}{\partial z}, y\frac{\partial}{\partial t}, (x)^2\frac{\partial}{\partial z} + (y)^2\frac{\partial}{\partial t}$. La m-dérivation $D = \mathrm{L}_X$ telle que $X = x\frac{\partial}{\partial x}$ vérifie $D\left((x)^2\frac{\partial}{\partial z} + (y)^2\frac{\partial}{\partial t}\right) \notin \mathfrak{P}$.

Remarque 3.14 Toute m-dérivation du normalisateur $\mathfrak{N}$ de $\mathfrak{P}$ est une somme de dérivée de Lie par rapport à un champ de vecteurs de son normalisateur, et d'une m-dérivation de degré -2. On sait que $\mathfrak{N}$ contient $\mathfrak{P}$, alors ce résultat découle du Théorème 3.6 et du Théorème 3.9.

Corollaire 3.15 *Toute m-dérivation de l'algèbre de Lie de tous les champs polynomiaux (resp. de l'algèbre de Lie affine $H_{-1} \oplus H_0$) de $\mathbb{R}^n$ est une m-dérivation intérieure. Toutes les m-dérivations des sous-algèbres de Lie des champs affines de la forme $H_{-1} \oplus \mathfrak{P}_0$ sur $\mathbb{R}^n$ sont des dérivées de Lie par rapport à un champ affine.*

Démonstration La première assertion vient du Corollaire 3.12 en utilisant le fait que l'algèbre de Lie de tous les champs polynomiaux de $\mathbb{R}^n$ contient tous les champs linéaires diagonaux. Comme le normalisateur d'une sous-algèbre de Lie des champs affines est inclus dans $H_{-1} \oplus H_0$, alors la suite de la démonstration découle du fait que la sous-algèbre ne contient pas de champs quadratiques et des Théorème 3.6, 3.9. □

On peut se demander s'il existe des m-dérivations non nulles de degré -2 compte tenu des nombreuses contraintes qu'elles doivent vérifier. L'exemple suivant où l'algèbre de Lie $\mathfrak{P}$ est séparée ou non séparée, nous éclaire sur ce point,

Exemple 3.16 La série centrale descendante de chacune des deux algèbres de Lie suivantes est constante égale à l'algèbre de Lie elle-même $\ominus$ l'algèbre de Lie engendrée par E.

Dans $\mathbb{R}^3$, soit la $\mathbb{R}-$algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, E, x\frac{\partial}{\partial z}, (x)^2\frac{\partial}{\partial z}$. L'application $\mathbb{R}-$linéaire D_0 définie par $D_0\left((x)^2\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}$ et nulle autrement, est une 3-dérivation non intérieure de champs de vecteurs.

La $\mathbb{R}-$algèbre de Lie engendrée par les champs de vecteurs polynomiaux $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, E, $x\frac{\partial}{\partial z} - y\frac{\partial}{\partial z}$, $xy\frac{\partial}{\partial z} - \frac{1}{2}(x)^2\frac{\partial}{\partial z} - \frac{1}{2}(y)^2\frac{\partial}{\partial z}$ admet une 3-dérivation non intérieure de champs de vecteurs définie par, l'application $\mathbb{R}-$linéaire D_1 telle que la seule image non nulle est

$$D_1\left(xy\frac{\partial}{\partial z} - \frac{1}{2}(x)^2\frac{\partial}{\partial z} - \frac{1}{2}(y)^2\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

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Journal of Generalized Lie Theory and Applications

Randriambololondrantomalala, J Generalized Lie Theory Appl 2015, S2
http://dx.doi.org/10.4172/1736-4337.S2-002

Research Article Open Access

The *m*-Derivations of Analytic Vector Fields Lie Algebras

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Abstract

We consider a (real or complex) analytic manifold *M*. Assuming that *F* is a ring of all analytic functions, full or truncated with respect to the local coordinates on *M*; we study the (m ≥ 2)-derivations of all involutive analytic distributions over *F* and their respective normalizers.

Keywords: *m*-derivations; Analytic vector fields Lie algebras; Distributions; Generalized foliations; Stein manifolds; Compact holomorphic manifolds; Chevalley-Eilenberg's cohomology; Compactly supported vector fields

Introduction and Preliminary

We know several embedding theorems in differential geometry, some of them are of John F. Nash in Riemannian manifolds [1,2], of Whitney [3] in differentiable manifolds and of Grauert in analytic manifolds cf. [4]. They make easy certain study on a differentiable manifold. Here, we are interested to a real or complex analytic *n*-dimensional manifold *M* and let *F*(*M*) be the ring of all analytic functions on *M*. We know that these manifolds can be considered as smooth manifolds. But certain property on a smooth manifold cannot be true on *M*, for example the global representation of a smooth function germ theorem. Grabowski had this problem when he studied derivations of the real or complex analytic vector fields Lie algebra cf. [5] and he used Stein manifolds to avoid technical difficulties in them. Here, we examine not only the derivations but the ($m \geq 2$)-derivations (generalization of derivation's notion) of a Lie subalgebra of the real or complex analytic vector fields Lie algebra on *M*, using Lie algebra tools. In advance, we state that the considered Lie algebras have enough sections more than constant ones in the Lie algebra of all analytic vector fields. Then, we consider only Stein spaces unless expressly stated in a complex analytic case. In the real analytic one, we don't need more hypothesis because of the imbedding theorem of Grauert and Cartan theorems [6]. More precisely, any real analytic manifold can be considered as a closed submanifold of a certain $\mathbb{R}^l$ (a " real Stein manifold"). Now, an m-derivation of a Lie algebra *A* is a linear map from *A* to itself which is distributive on the brackets $[X_1,[X_2,\ldots[X_{m-1},X_m]\ldots]]$, where all X_i are in *A*. On the one hand, we have studied the *m*-derivations of polynomial vector fields Lie algebras on $\mathbb{R}^n$ in studies of 7. Randriambololondrantomalala [7], an important Lie subalgebra of analytic vector fields, we found that Lie algebras of derivations are different to those of ($m > 2$)-derivations. One can see the following example, on $\mathbb{R}^3$, the Lie $\mathbb{R}$-algebra is spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}, y\frac{\partial}{\partial x}, (y)^2\frac{\partial}{\partial x}$ and let's define the $\mathbb{R}$–linear map *D* by $D\left((y)^2\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial x}$ which is zero otherwise. It's clear that *D* is not a derivation, but a 3-derivation. On the other hand, all *m*-derivations of a distribution over the full or truncated rings of smooth functions on a differentiable manifold in literature of Randriambololondrantomalala [8], are derivations. These facts lead us to ask if a distribution Lie algebra on an analytic manifold has results as the one or the other above results. So, we will divide our paper into three parts. First, we take a real or complex analytic involutive distribution Ω over *M*. That is to say, Ω is a *F*(*M*)-submodule of the analytic vector fields Lie algebra $\chi(M)$ on *M*. We can find some examples of these distributions and the interests for studying their derivations in literature of Grabowski and Cartan [5,6]. Here, we find the Ω's centralizer and the derivative ideal of Ω. We can say also that the normalizer of Ω is a Lie subalgebra of analytic vector fields. In addition, we find out that all *m*-derivations of Ω (resp. of the normalizer of Ω) are inner with respect to a normalizer's vector field (resp. are inner). Second, assuming that Ω is an involutive distribution on *M* over a subring *F* of *F*(*M*), namely an *F*-submodule of $\chi(M)$ stable by the vector fields bracket, where *F*≠ *F*(*M*). One can consider a system of commuting vector fields on *M* as in studies of Randriambololondrantomalala [8] and all distribution Lie subalgebras of the Lie algebra of analytic vector fields which commute with this system. The normalizer of Ω is an analytic vector fields Lie algebra and contains locally all constant vector fields and Euler's vector field. But in general, we can't use the reasoning by Randriambololondrantomalala [7] to characterize *m*-derivations of Ω. We make more explicit all *m*-derivations of Ω and of some of its normalizer. Whereas, in the end, we discuss the Lie algebras of holomorphic vector fields, especially when the holomorphic manifold is not a Stein one, and Lie algebras of locally polynomial vector fields on an analytic manifold *M*. Their *m*-derivations as well as their normalizers can be characterized by using some results of Randriambololondrantomalala [7].

Therefore, we have found the *m*-derivations of all distributions over a set of full or truncated analytic functions with respect to the local coordinates on *M*. When $m = 2$, we deduce from our results some [5]'s theorems. Third, we can apply our theorems on Lie algebras of real or complex analytic vector fields on *M*, of generalized foliation on *M* cf. [9], a Lie subalgebra of analytic vector fields on $\mathbb{S}^2$ and on $\mathbb{T}^2$, Riemann surfaces, etc. Relations between the Lie algebra of compactly supported vector fields and the compactness of *M* are discussed. Moreover, we emphasize the extensions of our theorems when the studied distributions are singular, by using the complexification of a real analytic manifold, Hartogs and Riemann extension theorems. Of course, in these circumstances, we can use theory of coherent sheaves

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Received November 23, 2015; **Accepted** December 17, 2015; **Published** December 19, 2015

Citation: Randriambololondrantomalala P (2015) The *m*-Derivations of Analytic Vector Fields Lie Algebras. J Generalized Lie Theory Appl S2: 002. doi:10.4172/1736-4337.S2-002

made by Cartan [6] in a Stein case or pass into Grabowski's conjecture cf. [9]. We interpret our results in Chevalley-Eilenberg cohomology sense when $m = 2$.

Following the above notations, let M be a real or complex analytic n-dimensional manifold. In complex case, we regard a Stein manifold unless special mention. We denote by $\chi(M)$ the Lie algebra of analytic vector fields on M and $F(M)$ the ring of analytic functions on M. Throughout this paper, we take an atlas in which every chart are connected. Then, the open subset of a chart U where a non-trivial subset of $\chi(M)$ doesn't vanish, is dense on U (non-trivial means different to $\{0\}$). We can use certain results of Randriambololondrantomalala [7,8] because in the proofs of theorem of these papers we consider only analytic functions (polynomials, exponentials). In the same way, we don't need partition of the unity to make global some local results cf. [10]. In all sections of this article, we set an integer $m \geq 2$, recall that D is an m-derivation of a Lie algebra A if for $(X_i)_{1\leq i\leq m} \subset A$, we get $D[X_1,[X_2,\ldots,[X_{m-1},X_m]\ldots]] = [D(X_1),[X_2,\ldots,[X_{m-1},X_m]\ldots]] + \ldots + [X_1,[X_2,\ldots,[X_{m-1},D(X_m)]\ldots]]$. This D is said inner on a Lie algebra $\mathbb{B}$ containing A, if D is a Lie derivative with respect to an element of $\mathbb{B}$. Recall us another basic definition cf. [11].

Definition 1.1. A complex manifold M is a Stein manifold, if we have simultaneously the three following conditions: For every $x \neq y$, both in M; there is a holomorphic function f over M such that $f(x) \neq f(y)$. For all $x \in M$, it exists n holomorphic functions (f_i) over M such that df_i are linearly independent over $\mathbb{C}$ on x. If K is a compact set of M, the following set is compact (holomorphic convexity of M)

$$\left\{x \in M \,/\, |f(x)| \leq \sup_{y\in K} |f(y)|, \text{for all holomorphic functions } f \text{ over } M\right\}.$$

From these assertions, every local ring of holomorphic functions around $x \in M$ is spanned by holomorphic functions on M cf. [12].

Some results of the Lie algebra of compactly supported vector fields C_c relative to a Stein manifold are the following,

Proposition 1.2. *A complex analytic manifold M is compact iff C_c is non trivial, particularly if M is a Stein holomorphic manifold, C_c is trivial.*

Proof. It's obvious that if M is compact, then $\mathfrak{C}_c = \chi(M)$ and C_c is not trivial. Conversely, suppose that M isn't a compact set and there is $X \in C_c$ such that $K = \mathrm{Supp}(X) \neq \emptyset$. We can consider $K \neq M$ because M is not compact. Then, we have the nullity of X in the open set $\complement K \neq \emptyset$. By analyticity, X vanishes in whole M. Hence, we have a contradiction about $K \neq \emptyset$ and we obtain M is a compact set. It's clear that a Stein space is never a compact set by definition, then its Lie algebra of compactly supported vector fields is trivial.

The *m*-derivations defined by distributions on *F*(*M*)

Let Ω be a non-trivial involutive analytic distribution over the analytic functions ring on M. Let N be the normalizer of Ω in $\chi(M)$, that is to say that the set of all $X \in \chi(M)$ such that $[X,\Omega] \subset \Omega$, and $\mathfrak{B} = \{x \in M \,/\, \Omega(x) \neq \{0\}\}$. We can choose a connected domain U_i of a chart. If we suppose that it exists an open set O_i in U_i where Ω vanishes, then $\Omega_{|U_i} = \{0\}$ by analyticity. Otherwise, every open set in U_i contains an element of **B**. So, $\mathbf{B} \cap U_i$ is dense over U_i. Moreover, the collection of U_i forms an atlas of M, then **B** is dense over M. The set **B** is an open analytic submanifold of M. Particularly, **B** is a Stein cf. [13]. Thus, every vector field defined over **B** admits a continuous extension on M, and if this last one is analytic, then it's necessarily an element of the normalizer of Ω. We use this last fact when we deal with extension theorems.

We know by literature of Nagano's [14] result that Ω is integrable, then it yields a generalized foliation F on M cf. [10]. So, Ω is the Lie algebra of tangent vector fields to the foliation and L_F the one of all foliation preserving vector fields. It is known that the normalizer N in $\chi(M)$ of Ω contains L_F cf. [10]. Hence, the restriction of the foliation in **B** is non singular.

Proposition 2.1. *The centralizer of Ω vanishes and the derivative ideal of Ω coincides with Ω itself.*

Proof. We say that $X \in \chi(M)$ is in the centralizer of Ω if $[X, \Omega]=\{0\}$; and the derivative ideal of denoted by $[\Omega, \Omega]$ is the Lie algebra spanned by all brackets of two elements of Ω. Suppose there is an non zero element X of the centralizer, we have $[X, f\Omega] = (X(f))\Omega = \{0\}$, for all $f \in F(M)$. It's not possible in a Stein manifold or in a real analytic manifold if X doesn't vanish identically over M and if $\Omega \neq \{0\}$. Along with this result, we can adapt the proof of Proposition 2.28 of studies of Randriambololondrantomalala [15] and assert that $[\Omega, \Omega] = \Omega$.

Let's recall an Hartogs's extension theorem and Riemann extension theorem.

Theorem 2.2. *(Hartogs [16]) Let be $t \geq 2$ and D be a bounded domain in $\mathbb{C}^t$. In addition, K be a compact subset of D such that $D - K$ is a connected set. Then all holomorphic functions f over $D - K$ can be extended holomorphically to D.*

Theorem 2.3. *(Riemann extension theorem) Let U be an open set in $\mathbb{C}$ and $z_0 \in U$. If $f : U - \{z_0\} \to \mathbb{C}$ is holomorphic function such that z_0 is a removable singularity of f, then f can be extended into an unique holomorphic function $\bar{f}$ in U where $\bar{f}(z_0) = \lim_{z\to z_0} f(z)$.*

Theorem 2.4. *In holomorphic case, all m-derivations of Ω, L_F, and of* N *are Lie derivatives with respect to elements belonging to* N. *In real analytic one, we have the same results if* **B** $= M$.

Proof. We can prove this assertion over **B** by Theorem 2.1 of studies of Randriambololondrantomalala [8] using Proposition 2.1 and partially Theorem 3.7 of literature of Randriambololondrantomalala [10]. For the corresponding extension theorem over M, we adopt the following arguments. We know that **B** is dense over M, then the restriction of **B** in each domain of a chart U is dense over U (U is a bounded set). The complement of this $\mathbf{B}\cap U$ in U can be considered as a compact set of the chart such that $\mathbf{B}\cap U$ is connected. In holomorphic case, when $n \geq 2$, we use Hartogs's theorem in a domain of the chart, so the extension theorem over M holds. If $n = 1$, we know by the isolated zeros principle that the domain of chart contains only a finite number of zeros in the corresponding restriction of **B**. By continuity at these zeros, which are removable singularities, the Riemann extension theorem can be used. Of course, if **B** $= M$ in real analytic situation, the extension theorem is applicable.

Definition 2.5. The complexification of a real analytic manifold M is a holomorphic manifold $\mathbb{H}$ such that there is a real analytic embedding f: $M \to \mathbb{H}$ where $\mathbb{H}$ has a holomorphic atlas $(U_i, \varphi_i)_i$ and $\varphi_i(f(M) \cap U_i) = \varphi_i(U_i) \cap \mathbb{R}^n$. We have a Stein complexification if $\mathbb{H}$ is Stein.

The next theorem is due by Grauert cf. [4,12].

Theorem 2.6. *Every real analytic manifold has a Stein complexification and can be analytically properly embedded into an Euclidean space $\mathbb{R}^N$.*

The following complexification of a Lie subalgebra $\mathbb{G}$ of the real analytic vector fields Lie algebra of M is in the following sense: if M can be embedded in a holomorphic manifold $\mathbb{H}$, the complexification $\overline{\mathbb{G}}$ of $\mathbb{G}$ in $\mathbb{H}$ is such that $\overline{\mathbb{G}}_{|M} = \mathbb{G}$.

Theorem 2.7. *If the complexification of* Ω *in a Stein space T is still an involutive distribution, then the first result of the Theorem 2.4 holds in real analytic case.*

Proof. We use the complexification of M on a Stein space T. Consequently, let be $\overline{\Omega}$ the complexification over T of Ω. Recall that $\overline{\Omega}$ is an involutive distribution over T where its normalizer on $\chi(T)$ is denoted by N_0. So, all m-derivations of Ω, of L_F, and of N have their complexified m-derivations over T on respectively $\overline{\Omega}$, the Lie algebra of all foliation preserving vector fields $\overline{L_{\mathfrak{F}}}$ on T and N_0. By the results of Theorem 2.4, these last m-derivations are Lie derivatives with respect to elements belonging to N_0. We can affirm that $\mathfrak{N}_{0|M} = \mathfrak{N}$ and $\overline{L_{\mathfrak{F}}}_{|M} = L_{\mathfrak{F}}$ by $\overline{\Omega}_{|M} = \Omega$. Thus, we have the same result as in the first part of Theorem 2.4.

By definition, the first space of Chevalley-Eilenberg's cohomology of a Lie algebra $\mathcal{A}$ denoted by $H^1(\mathcal{A})$ is $\mathrm{Der}(\mathcal{A})/\mathrm{ad}_{\mathcal{A}}$ with $\mathrm{Der}(\mathcal{A})$ the Lie algebra of all derivations of $\mathcal{A}$ and $\mathrm{ad}_{\mathcal{A}}$ the set of inner derivations of $\mathcal{A}$.

Throughout this paper, we suppose that all hypothesis of Theorem 2.7 are satisfied or $\mathfrak{B} = M$, in real analytic case.

Following ideas of Theorem 3.7, Corollary 2.14 and Remark 2.15 of literature of Randriambololondrantomalala [10], we state

Corollary 2.8. *The first space of Chevalley-Eilenberg's cohomology of* Ω, $L_{\mathfrak{F}}$, *and of* $\mathfrak{N}$ *is respectively isomorphic to the following respective Lie algebras,* $\mathfrak{N}/\Omega$, $\mathfrak{N}/L_{\mathfrak{F}}$, $\{0\}$.

Remark 2.9. *By Theorem 2.4, we deduce Theorem 3.2 and 4.1 of studies of Grabowski* [5].

The *m*-derivations associated to a distribution over a subring of *F*(*M*)

Let be an atlas of M such that Ω is locally spanned by $\left(\frac{\partial}{\partial x^i}\right)_{1\le i\le n}$ over the ring F_0 of real or complex functions depending only on $(x^j)_{1\le j\le k}$ with respect to the atlas (where $1 \le k < n$). We can consider Ω to be a Lie algebra which commutes with a system S of commuting vector fields by the usual bracket. That is to say, $S = \{X_1,\ldots,X_q\}$ such that $[X_i, X_j] = 0$ and S is locally of rank $n - k$ $(0 < q \le n)$. It is easy to check that $[\Omega,\Omega] = \Omega$ because of Randriambololondrantomalala's [8] result. So with the same reason, every m-derivation of Ω is local. Moreover, the normalizer N of Ω is locally isomorphic to $\Omega \oplus gl(n-k,\mathbb{R} \text{ or } \mathbb{C})$ as a vector space. We consider the closed 1–differential forms α^i and ω^i over a $(n - k)$-dimensional distinguished connected chart of the generalized foliation generated by S, where $i = k+1,\ldots,n$ and an m-derivation of Ω, $D^{(\alpha,\omega)} = \sum_{k+1\le j\le n}(\alpha^j + \omega^j)\otimes\frac{\partial}{\partial x^j}$ such that $ker(\omega^j) \supset F_0(U)\left\langle\frac{\partial}{\partial x^i}\right\rangle_{1\le i\le k}$ and $ker(\alpha^j) \supset F_0(U)\left\langle\frac{\partial}{\partial x^i}\right\rangle_{k+1\le i\le n}$ (S in this chart is $\{\frac{\partial}{\partial x^j}\}_{k+1\le j\le n}$, $F\langle A\rangle$ denotes a module spanned by A over a ring F) cf. [8]. We have omitted all singular charts of the foliation because the open set R of all regular points is dense over M cf. [10], we have no problem for the extension of our results from R towards M as in the previous section. By adapting Theorem 3.12 of literature of Randriambololondrantomalala [8], we obtain easily

Theorem 3.1. *All m-derivations of* Ω *(resp. of* $\mathfrak{N}$*) are a sum of a Lie derivative with respect to one element of* $\mathfrak{N}$ *and a derivation* $D^{(\alpha,w)}$ *as denoted before (resp. are similar to m-derivations of* Ω*).*

Hence, adopting the arguments of Theorem 3.19 of studies of Ravelonirina [17], we hold the following

Corollary 3.2. *When the rank of S is a positive constant n − k, the first spaces of Chevalley-Eilenberg's cohomology of* Ω *and of* N *are both isomorphic to* $(H_R(B))^{n-k} \times \left(\mathbb{R}^{(n-k)^2+n-k} \text{ or } \mathbb{C}^{(n-k)^2+n-k}\right)$ *with* $H_R(B)$ *is the de Rham cohomology of foliation basic forms of M.*

As we know, we can split the above Ω into a semi-direct sum of Lie algebras Ω_S^1 and Ω_S^2 as in studies of Randriambololondrantomalala [8], where they are modules on the ring $F_0(M)$ of constant functions over the leaves relative to the above generalized foliation. We notice that Ω_S^2 is spanned by S on $F_0(M)$. We can reason on a distinguished chart U with the coordinates $(x^i)_{1\le i\le n}$. The $F_0(U)$ is the set of all analytic functions depending only on $(x^i)_{1\le i\le k}$, $\Omega_{S|U}^1$ is spanned by $\left(\frac{\partial}{\partial x^i}\right)_{1\le i\le k}$ and $\Omega_{S|U}^2$ by $\left(\frac{\partial}{\partial x^i}\right)_{k+1\le i\le n}$ on $F_0(U)$.

Now, we discuss the m-derivations of Ω_S^1. The normalizer N^1 of this Lie algebra can be written as a direct sum of Lie algebras $\mathfrak{N}^1 = [\Omega_S^1,\Omega_S^1] \oplus [\mathcal{Q},\mathcal{Q}]$, where $\mathcal{Q}$ is the centralizer of Ω_S^1 in $\chi(M)$ and the center of Ω_S^1 is zero ($\mathcal{Q}$ is locally spanned by $\left(\frac{\partial}{\partial x^i}\right)_{k+1\le i\le n}$ on the ring of all analytic functions depending only on $(x^i)_{k+1\le i\le n}$). By a similar argument of Nakanishi [18], we deduce that all m-derivations of N^1 are a direct sum of those of Ω_S^1 and of $\mathcal{Q}$. By Theorem 2.4, it's clear that

Theorem 3.3. *Each m-derivation of* Ω_S^1 *(resp. of* $\mathfrak{N}^1$*) is a Lie derivative with respect to an element of* N^1.

The normalizer of Ω_S^2 is locally the sum of the $F_0(U)$-module spanned by $\underset{1\le i\le n}{\frac{\partial}{\partial x^i}}$ and a vector space generated by $\underset{k+1\le i,\, l\le n}{x^l\frac{\partial}{\partial x^i}}$. That is to say, its normalizer is $\mathfrak{N}$. In addition, its centralizer is Ω_S^2 itself. Because of $[\Omega_S^2,\Omega_S^2] = \{0\}$ or Ω_S^2 is nilpotent of order 1, we obtain easily

Theorem 3.4. *Every endomorphism of* Ω_S^2 *is an m-derivation of* Ω_S^2.

So, it's immediate that

Corollary 3.5. *The first space of Chevalley-Eilenberg's cohomology of* Ω_S^1, Ω_S^2 *and of* N^1 *are respectively isomorphic to the following respective Lie algebras,* $\mathfrak{N}^1/\Omega_S^1$, $\mathrm{End}(\Omega_S^2)/\Omega_S^2$, $\{0\}$.

Let's consider Ω_S^3 the Lie subalgebra of Ω, spanned by X_i over a ring $F \subseteq F(M)$. That is to say, F is locally the set of all analytic functions depending only on $(x^i)_{l\le i\le n}$ where $0 < l < k + 1$ (resp. $k + 1 < l < n + 1$). When $F = F(M)$ (resp. $F = F_0(M)$), it is a special case of Lie algebras defined in Theorem 2.4 (resp. in Theorem 3.1) when the submodule is generated by X_i. In the distinguished local coordinates, Ω_S^3 is spanned by $\left(\frac{\partial}{\partial x^j}\right)_{k+1\le j\le n}$ over F. The normalizer $\mathfrak{N}^3$ of Ω_S^3 coincides with the sum of Ω_S^3 and Ω_S^4 where the element of this last one is locally the following analytic vector fields $\sum_{l\le i\le k} f^i(x^t, l\le t\le k)\frac{\partial}{\partial x^i} + \sum_{1\le j\le l-1} g^j(x^t, 1\le t\le k)\frac{\partial}{\partial x^j}$

Citation: Randriambololondrantomalala P (2015) The *m*-Derivations of Analytic Vector Fields Lie Algebras. J Generalized Lie Theory Appl S2: 002. doi:10.4172/1736-4337.S2-002

$\left(\text{resp.} \underset{1\le i\le k}{f^i(x^t, 1\le t\le k)\frac{\partial}{\partial x^i}} + \underset{k+1\le j\le l-1}{h^j(x^t, l\le t\le n)x^j\frac{\partial}{\partial x^j}}\right)$. In the first case, we can adapt Theorem 2.4 because all analytic functions depending on (x^j) where $k+1\le j\le n$, are in the base ring of Ω_S^3. In the following case, it is easy to see that Theorem 3.1 can be adapted to Ω_S^3. Thus

Theorem 3.6. *In the first case, every m-derivation D of Ω_S^3 is a Lie derivative with respect to a $\mathfrak{N}^3$'s element; in the second, it is a sum of a Lie derivative of an element of $\mathfrak{N}^3$ and a $D^{(\alpha,w)}$ analogous to that of Theorem 3.1. In addition, the corresponding extension theorems hold.*

Corollary 3.7. *The first space of Chevalley-Eilenberg's cohomology of Ω_S^3 is respectively isomorphic to Ω_S^4 in the first case;*

$$\Omega_S^4 \oplus \left(\left(\mathrm{H}_R(\mathrm{B}')\right)^{l-k-1} \times \left(\mathbb{R}^{(l-k)(l-k-1)} \text{ ou } \mathbb{C}^{(l-k)(l-k-1)}\right)\right)$$

in the other one if S has a constant rank ($\oplus$ is a module direct sum and B' is the set of the corresponding foliation basic forms of M).

When we regard all the above normalizers on a distinguished chart, they contain locally all constant fields and Euler vector field. So, we ask one question: could we adapt Theorem 3.6 and Theorem 3.9 in [7] to these normalizers? The following remark shows us that this argument is false.

Remark 3.8. On $\mathbb{R}^3$, we consider the Lie $\mathbb{R}$-algebra A spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}, y\frac{\partial}{\partial x}, (y)^2\frac{\partial}{\partial x}, e^y\frac{\partial}{\partial x}, ye^y\frac{\partial}{\partial x}, y^2e^y\frac{\partial}{\partial x}, \dots$. Lemma 2.3 of literature of Randriambololondrantomalala [10] is not true for A, so the arguments proposed in the proof of Theorem 3.6 of Princy [7] don't hold in this situation.

Whereas, let P be a Lie subalgebra consisting locally of polynomial vector fields in $\chi(M)$, where the Euler vector field and all constant vector fields are locally in P. Especially, M is not supposed to be a Stein in the holomorphic case. Let's recall a well known theorem,

Theorem 3.9. *(The maximum principle)* [12] *Let be M a connected holomorphic manifold and f a holomorphic function on M such that $|f(z)|\le|f(z_0)|$, where $z_0 \in M$ for all $z \in M$; then f is a constant function.*

One consequence of the maximum principle is the following, if the holomorphic manifold M is compact, every holomorphic function on M is constant in every connected component of M. We know that M is locally connected, then each function over M is locally constant. Therefore, it's clear that if M is a compact and connected holomorphic manifold, the ring of all holomorphic functions on M is the complex constant functions ring. It's confirm that results of the following theorem complete our study about an involutive analytic distribution when $F(M)$ is reduced to $\mathbb{C}$.

By adapting Randriambololondrantomalala's [7] theorems and taking account that the vector field found in the proof of Theorem 3.6 of Princy [7] is analytic, it follows that

Theorem 3.10. *When m is even, all m-derivations of $\mathfrak{P}$ (resp. of the normalizer $\mathcal{N}$ of $\mathfrak{P}$ in $\chi(M)$) are a Lie derivative with respect to one and only one vector field belonging to $\mathcal{N}$ (resp. to the normalizer of $\mathcal{N}$ in $\chi(M)$). If m is an odd number, they are sum of a Lie derivative with respect to one element of $\mathcal{N}$ (resp. of the normalizer of $\mathcal{N}$) and an m-derivation of local homogeneous degree -2 of $\mathfrak{P}$.*

So, taking into account: the vanishing of the centralizer of $\mathfrak{P}$ cf. [19] p.91; both the proofs of Theorem 2.12 of Ravelonirina [19], Corollary 3.12 of Randriambololondrantomalala [7] and Theorem 3.7 in literature of Randriambololondrantomalala [10], we obtain

Corollary 3.11. *The first space of Chevalley-Eilenberg's cohomology of $\mathfrak{P}$, of $\mathcal{N}$ and of* N *is respectively isomorphic to the following respective Lie algebras $\mathcal{N}/\mathfrak{P}$,* N */ $\mathcal{N}$, {0}, where* N *is the normalizer of $\mathcal{N}$.*

Illustrations

Some illustrations of our theorems are given in this section.

Example 4.1. It's clear that Theorem 2.4 works well on the Lie algebra of all analytic vector fields $\chi(M)$, that is to say, all m-derivations of $\chi(M)$ are Lie derivatives by elements of $\chi(M)$. We can define the Lie algebra of compactly supported real analytic vector fields C and this theorem holds for this last one. In particular, $\mathrm{H}^1(\mathfrak{C}) = \chi(M)/\mathfrak{C}$ for a non-trivial $\mathfrak{C}$. More, $\mathrm{H}^1(\mathfrak{C}) = \{0\}$ if and only if M is compact. Obviously, we can use the above cited theorem on the Lie algebras of vector fields relative to a generalized foliation over M. We can cite some well known Stein spaces, $\mathbb{C}^n$, an open poly-disc in $\mathbb{C}^n$, non-compact Riemann surfaces, ... and build our results in these.

Example 4.2. Let be $\mathbb{S}^2$ a holomorphic compact connected manifold. It's not a Stein manifold nor a submanifold of $\mathbb{C}^u$ for any u, it's a compact Riemann surface. We choose the modified stereographic coordinates over this manifold. The $\mathbb{S}^2$ has an atlas composed by two charts (U, z_1) and (V, z_2) with the overlap map $\psi(z) = z^{-1}$ in $U \cap V$. We remark that the Lie algebra $\mathbb{L}$ of vector fields on M spanned over $\mathbb{C}$ by Y_1, Y_2 and Y_3 is the one of all polynomial vector fields in $\mathbb{S}^2$, where

$$Y_1 : \begin{cases} \frac{\partial}{\partial z^1} & \text{in } U \\ -(z^2)^2\frac{\partial}{\partial z^2} & \text{in } V \end{cases}, \qquad Y_2 : \begin{cases} -(z_1)^2\frac{\partial}{\partial z^1} & \text{in } U \\ \frac{\partial}{\partial z^2} & \text{in } V \end{cases},$$

$$Y_3 : \begin{cases} -z^1\frac{\partial}{\partial z^1} & \text{in } U \\ z^2\frac{\partial}{\partial z^2} & \text{in } V \end{cases}$$

. By Theorem 3.10, all m-derivations of $\mathbb{L}$ are Lie derivatives with respect to a vector field in $\mathbb{L}$ itself. That is to say, if D is an m-derivation of $\mathbb{L}$ defined by $D(Y_1) = \alpha^i Y_i$, $D(Y_2) = \beta^i Y_i$, $D(Y_2) = \gamma^i Y_i$; we have $D = \mathrm{L}_{-\gamma^1 Y_1 + \gamma^2 Y_2 + \alpha^1 Y_3}$.

When we look at $\mathbb{S}^2$ as a real analytic manifold, we set the charts $(U, (x^1, x^2))$ and $(V, (y^1, y^2))$ with the overlap map

$$\phi(x^1, x^2) = \left(y^1 = \frac{x^1}{(x^1)^2 + (x^2)^2}, y^2 = \frac{x^2}{(x^1)^2 + (x^2)^2}\right).$$

We set the real analytic vector field $Y_3 : \begin{cases} -x^1\frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^2} & \text{in } U \\ y^1\frac{\partial}{\partial y^1} + y^2\frac{\partial}{\partial y^2} & \text{in } V \end{cases}$ on $\mathbb{S}^2$

and the Lie algebra A of real analytic vector fields which commute with Y_3. This $\mathfrak{A}$ consists of real analytic vector fields Y such that

$$Y : \begin{cases} F_1\left(\frac{x^2}{x^1}\right)x^1\frac{\partial}{\partial x^1} + F_2\left(\frac{x^1}{x^2}\right)x^2\frac{\partial}{\partial x^2} & \text{in } U \\ F_3\left(\frac{y^2}{y^1}\right)y^1\frac{\partial}{\partial y^1} + F_4\left(\frac{y^1}{y^2}\right)y^2\frac{\partial}{\partial y^2} & \text{in } V \end{cases}$$

where in the $U \cap V$,

$$F_3\left(\frac{y^2}{y^1}\right) = \frac{1}{(y^1)^2 + (y^2)^2}\left(\left(-2F_2\left(\frac{y^1}{y^2}\right) + F_1\left(\frac{y^2}{y^1}\right)\right)(y^2)^2 - F_1\left(\frac{y^2}{y^1}\right)(y^1)^2\right)$$ and

$$F_4\left(\frac{y^1}{y^2}\right)=\frac{1}{(y^1)^2+(y^2)^2}\left(\left(-2F_1\left(\frac{y^2}{y^1}\right)+F_2\left(\frac{y^1}{y^2}\right)\right)(y^1)^2-F_2\left(\frac{y^1}{y^2}\right)(y^2)^2\right)$$

with F_k are arbitrary convenient functions of one variable. So, we can apply all theorems in Section 3 to $\mathfrak{A}$. Particularly $\mathrm{H}^1(\mathbb{L})=\{0\}$ and $\mathrm{H}^1(\mathfrak{A})=H^1_R(B)\times\mathbb{R}^2$.

Example 4.3. Indeed, Theorem 3.10 can be applied to a polynomial vector fields Lie algebra on the real analytic manifold $\mathbb{R}^n$ or the Stein manifold $\mathbb{C}^n$ having the corresponding hypothesis.

Example 4.4. We set the Lie algebra $\mathbb{A}$ over the Stein manifold $\mathbb{C}^3$ spanned over $\mathbb{C}$ by $\frac{\partial}{\partial z^1},\frac{\partial}{\partial z^2},\frac{\partial}{\partial z^3},z^1\frac{\partial}{\partial z^1}+z^2\frac{\partial}{\partial z^2}+z^3\frac{\partial}{\partial z^3},z^2\frac{\partial}{\partial z^1},(z^2)^2\frac{\partial}{\partial z^1}$. The normalizer of $\mathbb{A}$ is $\mathcal{N}_0=\mathbb{A}+R$, where R is the space over $\mathbb{C}$ generated by $z^1\frac{\partial}{\partial z^1},z^2\frac{\partial}{\partial z^2},z^3\frac{\partial}{\partial z^3},z^2\frac{\partial}{\partial z^3},z^3\frac{\partial}{\partial z^1}$. It is permit to use Theorem 3.10 and when m is even, every m-derivation of $\mathbb{A}$ is inner on $\mathcal{N}_0$. If m is odd, the m-derivation is a sum of an inner derivation on $\mathcal{N}_0$ with a $\mathbb{C}$-linear map D defined by $D\left((z^2)^2\frac{\partial}{\partial z^1}\right)=\alpha\frac{\partial}{\partial z^1}$ which is zero otherwise *($\alpha \in \mathbb{C}$)*. Moreover, all m-derivations of $\mathcal{N}_0$ are inner for all $m \geq 2$. So, $\mathrm{H}^1(\mathbb{A})\cong R\ominus\left\langle z^1\frac{\partial}{\partial z^1}+z^2\frac{\partial}{\partial z^2}+z^3\frac{\partial}{\partial z^3}\right\rangle$ and $\mathrm{H}^1(\mathcal{N}_0)\cong\{0\}$.

Remark 4.5. In the following example, Theorem 3.10 cannot be applied. We take the 2-torus $\mathbb{T}^2=\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$, which is a holomorphic connected compact manifold cf. [20], it's not a Stein. All overlap maps are translations, that is to say, they are holomorphic. We can define globally the Lie algebra of all constant vector fields $\mathfrak{Q}$ on M and find that $\mathfrak{Q}$ is the Lie algebra of all holomorphic vector fields over M. All endomorphisms of each Lie subalgebra of $\mathfrak{Q}$, which is inevitably nilpotent of order 1, are m-derivations of this subalgebra. The normalizer of this subalgebra or its centralizer is the Lie algebra of all vector fields over $\mathbb{T}^2$. But $\mathrm{H}^1(\mathfrak{Q})\cong\mathrm{End}(\mathfrak{Q})/\mathrm{ad}_{\mathfrak{Q}}$ and $\mathrm{H}^1(\chi(M))\cong\mathrm{End}(\chi(M))/\mathrm{ad}_{\chi(M)}$ since $\mathrm{H}^1(\chi(M))=\{0\}$ in smooth cases.

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Citation: Randriambololondrantomalala P (2015) The *m*-Derivations of Analytic Vector Fields Lie Algebras. J Generalized Lie Theory Appl S2: 002. doi:10.4172/1736-4337.S2-002

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, University of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications

Randriambololondrantomalala, J Generalized Lie Theory Appl 2015, S2
http://dx.doi.org/10.4172/1736-4337.S2-e001

Editorial Open Access

A Meeting of Great Minds, Sophus Lie and John Nash throughout their Works

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Keywords: Lie; Nash; Differential geometry

It is well known that Marius Sophus Lie (1842-1899) and John Forbes Nash (1928-2015) are great mathematicians. Sophus Lie comes from Norway and John Nash from United States of America. Their stories have certain resemblances and remarkable relations. This editorial would emphasize some of them. When they have started their university studies, their respective first interests were not mathematics.

That is to say, Lie has been in Astronomy and Nash in Chemical Engineering. Whereas, when they worked on mathematics, the first had Lobatchevski award in 1897 and the second, Nobel prize 1994 and Abel award 2015 (Niels Abel is the uncle of the wife of Sophus Lie: Anna Birch). In addition, their contributions in geometry are considerable, particularly in differential equations. Lie worked on transformation groups relative to partial differential equations, in other words, on Lie groups and on special non-associative algebras named Lie algebras. Nash discovered an important isometrically embedding theorem for a C^k-Riemannian manifold into an Euclidian space (k=1,3,4…), by studying an undetermined partial differential equations. Now, the methods they used offer us an important tool for continuing researches in differential geometry and in other fields cf. [1-9]. Next, these exceptional persons have continued to put more efforts into their mathematics works, even if they had a serious health problem in the middle of their careers. We hope that several mathematicians continue to make profits from results of Lie and Nash for the mathematics's promotion.

This special issue "Recent Advances of Lie Theory in Differential Geometry, in memory of John Nash" honored both Sophus Lie and John Nash as well as their works. John Nash died recently with his wife in a car crash on May 23, 2015. We are grateful regarding their contributions in differential geometry, generally in mathematics.

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Received October 02, 2015; **Accepted** October 12, 2015; **Published** October 14, 2015

Citation: Randriambololondrantomalala P (2015) A Meeting of Great Minds, Sophus Lie and John Nash throughout their Works. J Generalized Lie Theory Appl S2: e001. doi:10.4172/1736-4337.S2-e001

Canad. Math. Bull. Vol. 58 (4), 2015 pp. 692–703
http://dx.doi.org/10.4153/CMB-2015-022-2

Sur les algèbres de Lie associées à une connexion

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Abstract. Let Γ be a connection on a smooth manifold M. In this paper we give some properties of Γ by studying the corresponding Lie algebras. In particular, we compute the first Chevalley–Eilenberg cohomology space of the horizontal vector fields Lie algebra on the tangent bundle of M, whose the corresponding Lie derivative of Γ is null, and of the horizontal nullity curvature space.

Résumé. Etant donné une connexion Γ sur une variété différentiable M, dans ce papier on se propose de donner quelques propriétés de Γ en étudiant les algèbres de Lie associées à cette connexion. En particulier, on calcule le premier espace de cohomologie de Chevalley–Eilenberg de la partie horizontale de l'algèbre de Lie des champs de vecteurs sur le fibré tangent de M dont la dérivée de Lie correspondante de Γ est nulle, et de l'espace de nullité horizontal de la courbure.

1 Introduction

Nous avons étudié dans [1–3] certains types de sous-algèbres de Lie de champs de vecteurs sur une variété différentiable M de classe C^∞. Dans ce papier, nous proposons quelques propriétés d'une connexion sur M en étudiant certaines algèbres de Lie qui lui sont attachées.

Soit Γ une structure presque-produit sur le fibré tangent TM, où Γ^2 est égal à l'identité. Cette structure est une connexion au sens de Grifone définie sur M, *cf.* [5]. La donnée d'une telle connexion réalise une décomposition de TTM du fibré tangent de TM en une somme directe d'espace horizontal $h(TM)$ et d'espace vertical $v(TM)$, où h et v sont respectivement le projecteur horizontal et le projecteur vertical de la connexion correspondant à la valeur propre respective 1 et −1. Soient $\mathcal{T}M$ le fibré tangent TM privé de la section nulle, et R la courbure de Γ. On s'interesse à l'algèbre de Lie $\mathfrak{A}_\Gamma$ des champs de vecteurs sur $\mathcal{T}M$ dont la dérivée de Lie correspondante de Γ est nulle, *cf.* [6], et à l'algèbre de Lie de l'espace de nullité horizontal de la courbure $\mathfrak{N}_R^h$. On étudie le premier espace de cohomologie de Chevalley–Eilenberg de l'espace de nullité horizontal de la courbure. On donne quelques propriétés de la connexion à partir du normalisateur de $\mathfrak{N}_R^h$ et à partir de l'espace de cohomologie de Chevalley–Eilenberg de $\mathfrak{A}_\Gamma$. En effet, on peut appliquer un résultat dans [2] pour le calcul cohomologique

Reçu par la rédaction le 8 août 2013.
Publié electronique au 7 septembre 2015.
Les auteurs sont supportés par une bourse de recherche, initiative de Pr. Ingrid Daubechies en collaboration avec "Institute for the Conservation of Tropical Environments" (ICTE) Madagascar.
Classification (AMS) par sujet: **17B66**, 53B15, 17B56.
Mots clés: algèbre de Lie, connexion, cohomologie de Chevalley–Eilenberg, champs dont la dérivée de Lie correspondante à une connexion est nulle, espace de nullité de la courbure.

de l'espace de nullité horizontal. En utilisant certaines propriétés d'une distribution de classe C^∞, on trouve que la partie horizontale du normalisateur de $\mathfrak{N}_R^h$ est égale à $\mathfrak{N}_R^h$ si et seulement si la connexion est plate. L'isomorphisme (en tant que module) entre le premier espace de cohomologie de Chevalley–Eilenberg de $\mathfrak{N}_R^h$ et la partie verticale du normalisateur de $\mathfrak{N}_R^h$ dans $\chi(\mathcal{T}M)$ est équivalente à la nullité de la courbure. L'intersection de $\mathfrak{A}_\Gamma$ avec l'espace de nullité de la courbure $\mathfrak{N}_R$ est un produit direct de l'algèbre de Lie $\mathfrak{A}_\Gamma^h$ des champs de $\mathfrak{A}_\Gamma$ dans l'espace horizontal et de l'algèbre de Lie $\mathfrak{A}_\Gamma^v$ des champs de $\mathfrak{A}_\Gamma$ dans l'espace vertical. En particulier, l'algèbre de Lie $\mathfrak{A}_\Gamma^h$ coïncide à l'intersection de la distribution $\mathfrak{N}_R$ avec l'ensemble des champs horizontaux projetables. Compte tenu du fait que $\mathfrak{A}_\Gamma^h$ est une distribution involutive de M, les résultats de [2] s'appliquent à $\mathfrak{A}_\Gamma^h$. Ainsi, l'idéal dérivé de $\mathfrak{A}_\Gamma^h$ est $\mathfrak{A}_\Gamma^h$ lui-même. Si pour tout $x \in \mathcal{T}M$, il existe $X \in \mathfrak{A}_\Gamma^h$ tel que $X(x) \neq 0$, alors le centre de $\mathfrak{A}_\Gamma^h$ est réduit à zéro. Alors le premier espace de cohomologie de Chevalley–Eilenberg de $\mathfrak{A}_\Gamma^h$ (resp. du normalisateur quotienté avec le centralisateur de $\mathfrak{A}_\Gamma^h$ dans $\chi(\mathcal{T}M)$) est égal au quotient de ce normalisateur avec $\mathfrak{A}_\Gamma^h$ (resp. à $\{0\}$). Par ailleurs, on trouve qu'un champ de vecteurs est un élément de $\mathfrak{A}_\Gamma$ si et seulement s'il laisse invariant les sous-espaces propres correspondants aux valeurs propres de Γ. La partie verticale $\mathfrak{A}_\Gamma^v$ est le centralisateur de l'algèbre de Lie engendré par tous les champs horizontaux et projetables. On en tire que le premier espace de cohomologie de Chevalley–Eilenberg de l'algèbre de Lie $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ est le produit direct des premiers espaces de cohomologie de Chevalley–Eilenberg de $\mathfrak{A}_\Gamma^h$ et de $\mathfrak{A}_\Gamma^v$. Cet espace de cohomologie est nul si et seulement si la connexion est plate. De même, on a l'équivalence entre la nullité de la courbure et la nullité de la partie verticale du centralisateur de $\mathfrak{A}_\Gamma^v$ dans $\chi(\mathcal{T}M)$.

2 Préliminaires

Dans toute la suite, M est une variété différentiable de dimension n, TM est le fibré tangent de M. On note $\mathcal{T}M$ le fibré TM privé de la section nulle. Tous les objets utilisés sont supposés C^∞ sur M ou sur $\mathcal{T}M$, sauf mention expresse. L'ensemble $\chi(M)$ (resp. $\chi(\mathcal{T}M)$) désigne l'algèbre de Lie des champs de vecteurs sur M (resp. sur $\mathcal{T}M$) avec le crochet habituel de champs de vecteurs. On notera dans la suite par $\cong$ un isomorphisme d'algèbres de Lie.

Un sous-ensemble S de $\chi(M)$ (resp. $\chi(\mathcal{T}M)$) vérifiant (H) signifie que pour tout $x \in M$ (resp. $x \in \mathcal{T}M$), il existe $X \in S$ tel que $X(x) \neq 0$. Une distribution sur M (resp. sur $\mathcal{T}M$) est un $F(M)$-sous-module de $\chi(M)$ (resp. $F(\mathcal{T}M)$-sous-module de $\chi(\mathcal{T}M)$). On adopte la convention d'Einstein sur la sommation d'indices, sauf mention expresse.

Définition 2.1 On a une suite exacte de fibrés vectoriels sur TM, *cf.* [5] :

$$0 \to \pi^*(TM) \xrightarrow{i} TTM \xrightarrow{j} \pi^*(TM) \to 0,$$

où $\pi\colon TM \to M$ la projection du fibré tangent à M ; $P\colon TTM \to TM$ la projection du fibré tangent à TM, i : l'injection naturelle ; $j = (P, \pi_*)$ où π_* est l'application linéaire tangente de π. L'application $J = i \circ j$ est la structure tangente sur TM.

Définition 2.2 On appelle connexion au sens de Grifone sur M, *cf.* [5], une 1-forme vectorielle Γ de TM, C^∞ sur $TM - \{0\}$ telle que $J\Gamma = J$, $\Gamma J = -J$.

On a l'égalité $\Gamma^2 = \mathrm{I}$ où I est l'application identité de $\chi(TM)$ et Γ a deux valeurs propres 1 et −1. La connexion ainsi définie est une structure presque-produit sur TM. Le projecteur horizontal (resp. le projecteur vertical) de Γ est défini par $h = \frac{1}{2}(\mathrm{I} + \Gamma)$ (resp. par $v = \frac{1}{2}(\mathrm{I} - \Gamma)$).

Une connexion Γ permet d'obtenir une décomposition de TTM, le fibré tangent de TM, en somme d'espaces horizontal et vertical :

$$TTM = H(TM) \oplus V(TM)$$

avec

$$H(TM) = \mathrm{Im}(h) = \mathrm{Ker}(v) \quad \text{et} \quad V(TM) = \mathrm{Im}(v) = \mathrm{Ker}(h).$$

Dans tout ce qui suit, Γ est une connexion au sens de Grifone.

Définition 2.3 La courbure de la connexion Γ est définie par la 2-forme vectorielle $R = -\frac{1}{2}[h,h]$ où

$$\frac{1}{2}[h,h](X,Y) = [hX,hY] + h[X,Y] - h[hX,Y] - h[X,hY], \quad \forall X, Y \in \chi(\mathcal{T}M).$$

En coordonnées locales (x^i, y^j) de TM, Γ s'écrit $dx^i \otimes \frac{\partial}{\partial x^i} - 2\Gamma_i^j dx^i \otimes \frac{\partial}{\partial y^j} - dy^i \otimes \frac{\partial}{\partial y^i}$, *cf.* [5] :

$$R = \frac{1}{2} R^i_{jk} dx^j \wedge dx^k \otimes \frac{\partial}{\partial y^i} \quad \text{où} \quad R^i_{jk} = \frac{\partial \Gamma^i_k}{\partial x^j} - \frac{\partial \Gamma^i_j}{\partial x^k} + \Gamma^l_k \frac{\partial \Gamma^i_j}{\partial y^l} - \Gamma^l_j \frac{\partial \Gamma^i_k}{\partial y^l}.$$

L'espace de nullité de la courbure de Γ est

$$\mathfrak{N}_R = \{X \in \chi(\mathcal{T}M) \text{ tel que } R(X,Y) = 0, \forall Y \in \chi(\mathcal{T}M)\}.$$

L'espace $\mathfrak{N}_R$ est une distribution de $\mathcal{T}M$. Comme la courbure R est semi-basique, l'espace vertical est inclus dans $\mathfrak{N}_R$. En général, l'espace de nullité $\mathfrak{N}_R$ n'est pas involutif, comme le montre l'exemple suivant.

Exemple 2.4 Soient $M = \mathbb{R}^3$, $(x^i, y^i)_{1 \le i \le 3}$ le système de coordonnées dans $T\mathbb{R}^3$ et Γ une connexion linéaire au sens de Grifone telle que $\Gamma^1_1 = x^2 y^3$, $\Gamma^1_3 = x^1 y^2$, et les autres nuls. Les coefficients de la courbure R sont $R^1_{2,1} = y^3$, $R^1_{3,1} = -y^2$, et les autres nuls.

L'espace de nullité de la courbure $\mathfrak{N}_R$ est

$$\mathfrak{N}_R(y^3 \neq 0) = \left\{ \frac{y^2}{y^3} X^3 \frac{\partial}{\partial x^2} + X^3 \frac{\partial}{\partial x^3} + Y^i \frac{\partial}{\partial y^i} \right\}$$

et

$$\mathfrak{N}_R(y^3 = 0) = \left\{ \left(y^2 \frac{\partial X^3}{\partial y^3} \frac{\partial}{\partial x^2} + X^3 \frac{\partial}{\partial x^3} + Y^i \frac{\partial}{\partial y^i} \right)(y^3 = 0) \right\},$$

où les X^i et Y^i sont des fonctions C^∞ de $\mathbb{R}^3$. Le crochet de champs de l'espace de nullité $\frac{\partial}{\partial y^2}$ et $y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3}$ est $\frac{\partial}{\partial x^2}$, n'appartenant pas à $\mathfrak{N}_R$.

Remarque 2.5 Si $n = 1$, compte tenu de l'antisymétrie de la courbure, la connexion est trivialement plate. De même, si $n = 2$ et si on suppose que $\mathfrak{N}_R^h \neq \{0\}$ alors la connexion est plate.

En tenant compte de la remarque précédente, on suppose dans toute la suite que $n > 2$.

3 Etude de quelques algèbres de Lie rattachées à l'espace horizontal

Définition 3.1 L'espace de nullité horizontal est défini par $\mathfrak{N}_R^h = \mathfrak{N}_R \cap h\big(\chi(\mathcal{T}M)\big)$.

Dans [8], on a montré que cet espace de nullité horizontal est involutif. Ici, on s'intéresse sur quelques algèbres de Lie qui sont liées à $\mathfrak{N}_R^h$.

Proposition 3.2 *S'il existe un sous-ensemble de l'algèbre de Lie des champs de vecteurs projetables engendrant la distribution $\mathfrak{N}_R^h$, alors $\mathfrak{N}_R$ est involutive.*

Démonstration Comme les $\mathfrak{N}_R^h$ et $v\big(\chi(\mathcal{T}M)\big)$ sont des distributions involutives et que le module $\mathfrak{N}_R = \mathfrak{N}_R^h \oplus v\big(\chi(\mathcal{T}M)\big)$, il suffit de vérifier que $\big[\mathfrak{N}_R^h, v\big(\chi(\mathcal{T}M)\big)\big]$ est une partie de $\mathfrak{N}_R$. C'est le cas, car cet ensemble est inclus dans $v\big(\chi(\mathcal{T}M)\big) \subset \mathfrak{N}_R$ par le fait que $\mathfrak{N}_R^h$ soit engendré par des champs projetables. ∎

Définition 3.3 Une dérivation D d'une $\mathbb{R}$-algèbre de Lie A est une application $\mathbb{R}$-linéaire de A dans A telle que $\forall X, Y \in A$, $D[X, Y] = [D(X), Y] + [X, D(Y)]$.

Définition 3.4 On appelle premier espace de cohomologie de Chevalley–Eilenberg d'une algèbre de Lie A, *cf.* [9], l'espace quotient $\mathrm{H}^1(A) = \mathrm{Der}(A)/\mathrm{ad}(A)$ où $\mathrm{Der}(A)$ (resp. $\mathrm{ad}(A)$) est l'ensemble des dérivations (resp. des dérivations intérieures) de A.

Théorème 3.5 *L'idéal dérivé de $\mathfrak{N}_R^h$ coïncide à $\mathfrak{N}_R^h$. Si l'algèbre de Lie $\mathfrak{N}_R^h$ vérifie (H), le centralisateur de $\mathfrak{N}_R^h$ est nul et, si on note $\mathcal{N}_R$ le normalisateur de $\mathfrak{N}_R^h$ dans $\chi(\mathcal{T}M)$, alors*

$$\mathrm{H}^1(\mathfrak{N}_R^h) \cong \mathcal{N}_R/\mathfrak{N}_R^h \quad et \quad \mathrm{H}^1(\mathcal{N}_R) \cong \{0\}.$$

La partie horizontale du normalisateur de $\mathfrak{N}_R^h$ coïncide avec $\mathfrak{N}_R^h$ si et seulement si la connexion est plate.

Démonstration L'algèbre de Lie $\mathfrak{N}_R^h$ est une distribution involutive de classe C^∞ de $\mathcal{T}M$, alors la première partie des résultats découlent directement de ceux de [2, pp. 140–141].

Si la connexion est plate, alors il existe une structure de feuilletage régulier $\mathfrak{F}$ de dimension n sur $\mathcal{T}M$, *cf.* [6, p. 6], telle que $\mathfrak{N}_R^h$ est égal à l'algèbre de Lie $L_{\mathfrak{F}}$ des champs de vecteurs tangents au feuilletage. Le normalisateur de $L_{\mathfrak{F}}$ coïncide à l'algèbre de Lie des champs infinitésimaux de $\mathfrak{F}$. Ainsi, en utilisant la propriété du feuilletage, la partie horizontale de ce normalisateur est $\mathfrak{N}_R^h$.

Rappelons qu'un point x d'une variété différentiable est un point régulier d'une distribution Δ s'il existe un voisinage de x tel que la restriction de Δ sur ce voisinage est régulière (ou de rang constant). Réciproquement, on suppose que la connexion est non plate. L'ensemble des points réguliers d'une distribution est un ouvert dense dans la variété où elle est définie. Comme $\mathcal{T}M$ est localement connexe et la courbure est non nulle, alors il existe un ensemble ouvert de $\mathcal{T}M$ où le rang de la distribution est une constante $0 < k \leq n-2$. Par suite, on peut trouver une carte adaptée de système de coordonées (x^i, y^i) où $\mathfrak{N}_R^h$ est $\{X^i \frac{\partial}{\partial x^i}\}_{1\leq i\leq k}$. Ainsi, le normalisateur sur cette carte contient l'ensemble des $X^j(x^i, i \neq 1,\dots,k; y^l)\frac{\partial}{\partial x^j}{}_{k+1\leq j\leq n} + Y^j(x^i, i\neq 1,\dots,k; y^l)\frac{\partial}{\partial y^j}$. Donc, $X^{j_0}(x^i, i \neq 1,\dots,k; y^l)\frac{\partial}{\partial x^{j_0}} - \Gamma_{j_0}^j X^{j_0}\frac{\partial}{\partial y^j}$ avec $j_0 \geq k+1$ sont des éléments de la partie horizontale du normalisateur de $\mathfrak{N}_R^h$ dans cet ouvert. Par suite, $h(\mathcal{N}_R)$ est différente de $\mathfrak{N}_R^h$. ■

En utilisant la décomposition de $\mathcal{N}_R$ en une somme directe de modules $h(\mathcal{N}_R)$ et $v(\mathcal{N}_R)$, on a le suivant.

Corollaire 3.6 *La connexion est plate si et seulement si le premier espace de cohomologie de Chevalley–Eilenberg de $\mathfrak{N}_R^h$ est isomorphe (en tant que module) à la partie verticale du normalisateur de $\mathfrak{N}_R^h$ dans $\chi(\mathcal{T}M)$.*

Remarque 3.7 Il est à noter que la partie horizontale $h(\mathcal{N}_R)$ du normalisateur de $\mathfrak{N}_R^h$ est différente de l'ensemble $\mathcal{N}_R \cap h\big(\chi(\mathcal{T}M)\big)$. S'il existe une carte où $\mathfrak{N}_R^h$ est de rang $k > 0$ et $\mathfrak{N}_R^h(\Gamma_{j_0}^j) = \{0\}$ pour un $k < j_0 \leq n$ et pour tout j, alors Γ est plate si et seulement si $\mathcal{N}_R \cap h\big(\chi(\mathcal{T}M)\big)$ coïncide à $\mathfrak{N}_R^h$. La démonstration de cette dernière assertion est la même que celle de la deuxième partie du théorème 3.5 mais avec quelques petites modifications.

Définition 3.8 On définit l'algèbre de Lie $\mathfrak{A}_\Gamma = \{X \in \chi(\mathcal{T}M)$ tel que $[X,\Gamma] = 0\}$ où $[X,\Gamma] = 0$ veut dire que $\Gamma[X,Y] = [X,\Gamma(Y)]$ pour tout $Y \in \chi(\mathcal{T}M)$. Localement, si $(x^i)_{i=1,\dots,2n}$ sont les coordonnées naturelles sur $\mathcal{T}M$, la connexion $\Gamma = \Gamma_i^j dx^i \otimes \frac{\partial}{\partial x^j}$, et $X = X^i\frac{\partial}{\partial x^i}$ appartient à A_Γ si et seulement s'il satisfait le système de $4n^2$ équations linéaires aux dérivées partielles, *cf.* [6] :

$$X^i\frac{\partial \Gamma_k^j}{\partial x^i} - \Gamma_i^j\frac{\partial X^i}{\partial x^k} + \Gamma_k^i\frac{\partial X^j}{\partial x^i} = 0.$$

On essaie de déterminer les dérivations de $\mathfrak{A}_\Gamma$ dont on examinera particulièrement par la suite sa partie horizontale.

Désignons par $\mathfrak{A}_\Gamma^h$ l'ensemble $\{X \in \mathfrak{A}_\Gamma$ tel que $h(X) = X\}$ et par $\mathfrak{A}_\Gamma^v$ celui des X appartenant à $\mathfrak{A}_\Gamma$ tel que $v(X) = X$. En utilisant directement la définition 3.8, on a $\mathfrak{A}_\Gamma = \mathfrak{A}_h$. La proposition s'en suit.

Proposition 3.9 *Tous les éléments de l'algèbre de Lie $\mathfrak{A}_\Gamma$ sont projetables.*

Démonstration Soit $X \in \mathfrak{A}_\Gamma$, d'après ce qui précède $\forall Y \in \chi(\mathcal{T}M)$, on a l'égalité $[X, hY] = h[X,Y]$. Comme h est semi-basique, alors il s'annule sur l'espace vertical, *cf.* [5] ; on obtient $h[X,JY] = [X,hJY] = 0$. Et par l'expression de J, $J[X,JY] = 0$ pour tout $Y \in \chi(\mathcal{T}M)$. Alors X est projetable, *cf.* [6]. ■

Proposition 3.10 *Les ensembles $\mathfrak{A}_\Gamma^h$ et $\mathfrak{A}_\Gamma^v$ sont des idéaux de $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ et de $\mathfrak{A}_\Gamma$. Ces idéaux forment un produit direct.*

Démonstration Soient $X \in \mathfrak{A}_\Gamma^h$ et $Y \in \mathfrak{A}_\Gamma$, $\Gamma[Y, X] = [Y, \Gamma X] = [Y, X]$. Donc $[Y, X] \in \mathfrak{A}_\Gamma^h$ et $\mathfrak{A}_\Gamma^h$ est un idéal de $\mathfrak{A}_\Gamma$. Par ailleurs, $\mathfrak{A}_\Gamma^h$ est contenu dans $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$. En effet, l'expression de R est

$$(3.1)\qquad R(X, Y) = [\Gamma X, \Gamma Y] - \Gamma[\Gamma X, Y] + \Gamma^2[X, Y] - \Gamma[X, \Gamma Y], \quad \forall Y \in \chi(\mathcal{T}M).$$

En utilisant $\Gamma[X, Y] = [X, \Gamma Y]$, on a $R(X, Y) = 0$. Ainsi, il est immédiat que l'algèbre de Lie $\mathfrak{A}_\Gamma^h$ est un idéal de $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$.

On raisonne de la même manière que précédemment pour $\mathfrak{A}_\Gamma^v$.

Soient $X \in \mathfrak{A}_\Gamma^h$ et $Y \in \mathfrak{A}_\Gamma^v$, alors $\Gamma[X, Y] = [X, \Gamma(Y)] = [\Gamma(X), Y]$. On en tire que $[X, Y] = -[X, Y]$, et $[X, Y] = 0$. D'où le résultat. ■

Remarque 3.11 On peut même montrer que $\mathfrak{A}_\Gamma^h$ est un idéal de tout sous-module de $\mathfrak{A}_\Gamma$ qui le contient.

Proposition 3.12 *On note par H^0 l'ensemble des champs horizontaux projetables, alors $\mathfrak{A}_\Gamma^h = H^0 \cap \mathfrak{N}_R$.*

Démonstration D'après la proposition 3.9 et la proposition 3.10, on peut montrer que $\mathfrak{A}_\Gamma^h \subset H^0 \cap \mathfrak{N}_R$.

Soit $X \in H^0 \cap \mathfrak{N}_R$, un champ $X \in \mathfrak{N}_R$ signifie d'après l'expression de R que $[hX, h] - h[X, h] = 0$. Or le champ X est projetable alors $[hX, h] = 0$. Donc le champ horizontal $hX \in \mathfrak{A}_h = \mathfrak{A}_\Gamma$. Comme X est horizontal alors $hX = X$. Ainsi $X \in \mathfrak{A}_\Gamma^h$ et $H^0 \cap \mathfrak{N}_R \subset \mathfrak{A}_\Gamma^h$. ■

Remarque 3.13 On constate d'après la proposition 3.12 que l'idéal horizontal $\mathfrak{A}_\Gamma^h$ est inclus dans l'espace de nullité horizontal, car il est l'intersection de l'espace de nullité horizontal et de l'ensemble des champs de vecteurs projetables. En général, $\mathfrak{A}_\Gamma^h$ n'est pas un idéal de $\mathfrak{N}_R^h$. L'algèbre de Lie peut être réduit à zéro, c'est le cas de celui de l'exemple 2.4.

Proposition 3.14 *On suppose que l'algèbre de Lie $\mathfrak{A}_\Gamma^h$ vérifie (H). On note par $\mathfrak{C}_\Gamma$ le centralisateur de $\mathfrak{A}_\Gamma^h$ dans $\chi(\mathcal{T}M)$. Alors la partie horizontale de ce centralisateur est nul et $\mathfrak{C}_\Gamma \cap \mathfrak{A}_\Gamma = \mathfrak{A}_\Gamma^v$.*

Démonstration Si X est dans ce centralisateur horizontal, alors $[X, \mathfrak{A}_\Gamma^h] = \{0\}$. D'après la proposition 3.12, l'algèbre de Lie $\mathfrak{A}_\Gamma^h$ est un $F(M)$-module ; pour tout $f \in F(M)$, $[X, f\mathfrak{A}_\Gamma^h] = X(f)\mathfrak{A}_\Gamma^h = \{0\}$. Par suite, $X(f) = 0$ car $\mathfrak{A}_\Gamma^h$ vérifie (H). Le champ X étant horizontal, alors $X = 0$. Par ailleurs, d'après ce qui précède, le centralisateur de $\mathfrak{A}_\Gamma^h$ dans $\mathfrak{A}_\Gamma$ est inclus dans sa partie verticale $\mathfrak{C}_\Gamma^v$. Donc $\mathfrak{C}_\Gamma^v \cap \mathfrak{A}_\Gamma$ est contenu dans $\mathfrak{A}_\Gamma^v$ car $\mathfrak{C}_\Gamma^v$ est vertical. D'après la proposition 3.10, $\mathfrak{A}_\Gamma^v$ commute avec $\mathfrak{A}_\Gamma^h$, alors $\mathfrak{A}_\Gamma^v \subset \mathfrak{C}_\Gamma \cap \mathfrak{A}_\Gamma = \mathfrak{C}_\Gamma^v \cap \mathfrak{A}_\Gamma$. ■

Définition 3.15 Le relèvement complet de $X \in \chi(M)$ sur le fibré TM est noté $\bar{X}$. En coordonnées locales sur TM, $(x^i, y^j)_{1\le i,j\le n}$ où $X = X^i \frac{\partial}{\partial x^i}$, on a

$$\bar{X} = X^i \frac{\partial}{\partial x^i} + y^j \frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial y^i} \quad \text{avec } i, j = 1, \dots, n.$$

Le relèvement complet d'une partie A de $\chi(M)$ est $\bar{A} = \{\bar{X} \text{ tel que } X \in A\}$.

Proposition 3.16 *Soient l'application $\Pi\colon \mathcal{T}M \to M$, le fibré des vecteurs non nul tangent à M et Π_* son application tangente, l'algèbre de Lie $\mathfrak{A}^h_\Gamma$ est isomorphe à $\Pi_*(\mathfrak{A}^h_\Gamma)$.*

Démonstration D'après la proposition 3.12, $\Pi_*[\mathfrak{A}^h_\Gamma, \mathfrak{A}^h_\Gamma] = [\Pi_*(\mathfrak{A}^h_\Gamma), \Pi_*(\mathfrak{A}^h_\Gamma)]$. De la relation $[\mathfrak{A}^h_\Gamma, \mathfrak{A}^h_\Gamma] \subset \mathfrak{A}^h_\Gamma$, on a $[\Pi_*(\mathfrak{A}^h_\Gamma), \Pi_*(\mathfrak{A}^h_\Gamma)] \subset \Pi_*(\mathfrak{A}^h_\Gamma)$.

En considérant l'application $f\colon \Pi_*(\mathfrak{A}^h_\Gamma) \to \overline{\Pi_*(\mathfrak{A}^h_\Gamma)}$ avec $f(X) = \bar{X}$, on peut vérifier que f est bijective.

On a $\overline{[X, Y]} = [\bar{X}, \bar{Y}] \forall X, Y \in \Pi_*(\mathfrak{A}^h_\Gamma)$, alors

$$\overline{[\Pi_*(\mathfrak{A}^h_\Gamma), \Pi_*(\mathfrak{A}^h_\Gamma)]} = [\overline{\Pi_*(\mathfrak{A}^h_\Gamma)}, \overline{\Pi_*(\mathfrak{A}^h_\Gamma)}].$$

Ce qui donne $[\overline{\Pi_*(\mathfrak{A}^h_\Gamma)}, \overline{\Pi_*(\mathfrak{A}^h_\Gamma)}] \subset \overline{\Pi_*(\mathfrak{A}^h_\Gamma)}$. Et $\overline{\Pi_*(\mathfrak{A}^h_\Gamma)}$ est une algèbre de Lie, et $\Pi_*(\mathfrak{A}^h_\Gamma)$ est isomorphe à $\overline{\Pi_*(\mathfrak{A}^h_\Gamma)}$.

Par suite, soit la restriction de h suivante $h\colon \overline{\Pi_*(H^0 \cap \mathfrak{N}_R)} \to \mathfrak{A}^h_\Gamma$. Il est facile de montrer qu'elle est bien définie et surjective. On a de même $\mathrm{Ker}(h) = 0$ sur $\overline{\Pi_*(H^0 \cap \mathfrak{N}_R)}$. Donc cette restriction de h est bijective.

Le champ de vecteurs $\bar{X}$ étant projetable ; d'après la preuve de la proposition 3.12 on a $[h\bar{X}, h] = 0$ et $h[\bar{X}, \bar{Y}] = h[\bar{X}, h\bar{Y}]$ pour tout $\bar{Y} \in \overline{\chi(M)}$. Ainsi R s'écrit

$$R(\bar{X}, \bar{Y}) = [h\bar{X}, h\bar{Y}] - h[\bar{X}, h\bar{Y}] = [h\bar{X}, h\bar{Y}] - h[\bar{X}, \bar{Y}].$$

Si $\bar{X}, \bar{Y} \in \overline{\Pi_*(\mathfrak{A}^h_\Gamma)}$, d'après ce qui précède $h\bar{X}, h\bar{Y} \in \mathfrak{A}^h_\Gamma$. Par conséquent, $R(\bar{X}, \bar{Y})$ est horizontal. Comme R est semi-basique alors $R(\bar{X}, \bar{Y}) = 0$ et $h[\bar{X}, \bar{Y}] = [h\bar{X}, h\bar{Y}]$.

On démontre alors que $\Pi_*(\mathfrak{A}^h_\Gamma)$ est isomorphe à $\mathfrak{A}^h_\Gamma$ en considérant l'application $h \circ f$. ■

Proposition 3.17 *On note par $\mathcal{N}_\Gamma$ le normalisateur de $\mathfrak{A}^h_\Gamma$ dans $\chi(\mathcal{T}M)$. On a désigné par $\mathcal{C}^v_\Gamma$ l'ensemble $\mathcal{C}_\Gamma \cap v\big(\chi(\mathcal{T}M)\big)$, on obtient $\mathcal{N}_\Gamma = h(\mathcal{N}_\Gamma) \oplus \mathcal{C}^v_\Gamma$ en tant que module.*

Démonstration Soient $X^1 \in \mathfrak{A}^h_\Gamma$ et $Y \in \mathcal{N}_\Gamma$. En décomposant Y sur les sous-espaces propres de Γ, on a $Y = Y^1 + Y^2$ avec $Y^1 \in h\big(\chi(\mathcal{T}M)\big)$, $Y^2 \in v\big(\chi(\mathcal{T}M)\big)$. Donc on obtient que $[Y^1, X^1] + [Y^2, X^1]$ appartient $\mathfrak{A}^h_\Gamma$. Par ailleurs, les égalités suivantes sont obtenues en utilisant directement la définition du champ de vecteurs X_1,

$$\Gamma[Y^1, X^1] = [\Gamma Y^1, X^1] = [Y^1, X^1]$$

et

$$\Gamma[Y^2, X^1] = [\Gamma Y^2, X^1] = -[Y^2, X^1].$$

On en déduit que

$$\Gamma([Y^1, X^1] + [Y^2, X^1]) = [Y^1, X^1] - [Y^2, X^1] \in \mathfrak{A}^h_\Gamma. \tag{3.2}$$

Alors les $[Y^1, X^1]$ et $[Y^2, X^1]$ sont dans $\mathfrak{A}_\Gamma^h$ et donc $Y^1, Y^2 \in \mathcal{N}_\Gamma$. Or l'image par Γ de $[Y^1, X^1] + [Y^2, X^1]$ est égal à $[Y^1, X^1] + [Y^2, X^1]$ par définition de $\mathfrak{A}_\Gamma^h$. Ainsi par la relation (3.2) on a $[Y^2, X^1] = 0$ et $Y^2 \in \mathcal{C}_\Gamma^v$. D'où le résultat. ∎

Proposition 3.18 *L'algèbre de Lie $\mathcal{C}_\Gamma^v$ est un idéal de $\mathcal{N}_\Gamma$. On note par $\mathcal{N}$ le normalisateur de $\Pi_*(\mathfrak{A}_\Gamma^h)$ dans $\chi(M)$, si l'algèbre de Lie $\mathfrak{A}_\Gamma^h$ vérifie (H) alors les éléments de $h(\mathcal{N}_\Gamma)$ sont projetables et*

$$\mathcal{N} \cong \mathcal{N}_\Gamma / \mathcal{C}_\Gamma^v.$$

Démonstration

- Le centralisateur $\mathcal{C}_\Gamma$ étant toujours un idéal de $\mathcal{N}_\Gamma$ *cf.* [4], on a une somme directe de modules $\mathcal{C}_\Gamma = \mathfrak{C}_\Gamma^h \oplus \mathcal{C}_\Gamma^v$. De la proposition 3.14, on trouve que $\mathcal{C}_\Gamma^v$ est un idéal de $\mathcal{N}_\Gamma$.
- Soient $Y \in \mathcal{N}_\Gamma$ et $\mathfrak{A}_\Gamma^h$ vérifiant (H). D'après le théorème de Fröbenius et la proposition 3.12, il existe en tout point de $\mathcal{T}M$ une carte qui le contient, de système de coordonnées (x^i, y^j) tels que $Y(F(x^i)) = Y^j \frac{\partial F(x^i)}{\partial x^j} \subset F(x^i)$ et $Y^j \in F(x^i)$, où $F(x^i)$ l'ensemble des fonctions qui ne dépendent que des x^i. Donc tous les éléments de $\mathcal{N}_\Gamma$ sont projetables, ainsi que ceux de $h(\mathcal{N}_\Gamma)$. En utilisant la proposition 3.17, on a $\mathcal{N}_\Gamma = h(\mathcal{N}_\Gamma) \oplus \mathcal{C}_\Gamma^v$. Alors la restriction de Π_*, $\Pi_*: \mathcal{N}_\Gamma \to \Pi_*(\mathcal{N}_\Gamma)$ est un homomorphisme d'algèbres et a pour noyau $\mathcal{C}_\Gamma^v$. Donc, l'application $\Pi'_*: \mathcal{N}_\Gamma/\mathcal{C}_\Gamma^v \to \Pi_*(\mathcal{N}_\Gamma)$ est un isomorphisme d'algèbres de Lie.

Montrons que $\Pi_*(\mathcal{N}_\Gamma) = \mathcal{N}$. Si $X \in \mathcal{N}_\Gamma$ alors $\Pi_*[X, \mathfrak{A}_\Gamma^h] \subset \Pi_*(\mathfrak{A}_\Gamma^h)$. D'après la proposition 3.12, $[\Pi_*(X), \Pi_*(\mathfrak{A}_\Gamma^h)]$ est inclus dans $\Pi_*(\mathfrak{A}_\Gamma^h)$. Donc, on en déduit que $\Pi_*(\mathcal{N}_\Gamma) \subset \mathcal{N}$.

Réciproquement, si $X \in \mathcal{N}$ alors de la proposition 3.16, $[\bar{X}, \overline{\Pi_*(\mathfrak{A}_\Gamma^h)}] \subset \overline{\Pi_*(\mathfrak{A}_\Gamma^h)}$. De même, on obtient $h[\bar{X}, \overline{\Pi_*(\mathfrak{A}_\Gamma^h)}] \subset \mathfrak{A}_\Gamma^h$. Or $\bar{X}$ est projetable, par la preuve de la proposition 3.16, on a

$$h[\bar{X}, \overline{\Pi_*(\mathfrak{A}_\Gamma^h)}] = h\left[\bar{X}, h\left(\overline{\Pi_*(\mathfrak{A}_\Gamma^h)}\right)\right] = h[\bar{X}, \mathfrak{A}_\Gamma^h] \subset \mathfrak{A}_\Gamma^h.$$

Or $\mathfrak{A}_\Gamma^h \subset \mathfrak{A}_h$ alors $h[\bar{X}, \mathfrak{A}_\Gamma^h] = [h\bar{X}, \mathfrak{A}_\Gamma^h]$. Ainsi $[h\bar{X}, \mathfrak{A}_\Gamma^h] \subset \mathfrak{A}_\Gamma^h$ et $h\bar{X} \in \mathcal{N}_\Gamma$ avec $\Pi_*(h\bar{X}) = X \in \mathcal{N}$. Donc $X \in \Pi_*(\mathcal{N}_\Gamma)$ et $\mathcal{N} \subset \Pi_*(\mathcal{N}_\Gamma)$. D'où $\Pi_*(\mathcal{N}_\Gamma)$ est égal à $\mathcal{N}$. ∎

Théorème 3.19 ([2]) *Une distribution Ω de M est involutive si et seulement si $[\Omega, \Omega] = \Omega$.*

Démonstration Immédiate en utilisant la proposition 2.9 de [2, p. 141]. ∎

Théorème 3.20

(i) *L'algèbre de Lie $\mathfrak{A}_\Gamma^h$ (resp. $\mathcal{C}_\Gamma$) est un idéal caractéristique de $\mathfrak{A}_\Gamma$ et de $\mathcal{N}_\Gamma$ (resp. de $\mathcal{N}_\Gamma$). Plus précisement, on a $[\mathfrak{A}_\Gamma^h, \mathfrak{A}_\Gamma^h] = \mathfrak{A}_\Gamma^h$.*

(ii) *Si l'ensemble $\mathfrak{A}_\Gamma^h$ vérifie (H), alors l'algèbre de Lie des dérivations de $\mathfrak{A}_\Gamma^h$ correspond à la représentation adjointe de son normalisateur $\mathcal{N}_\Gamma$ et les premiers espaces de cohomologie de Chevalley–Eilenberg sont :*

$$\mathrm{H}^1(\mathfrak{A}_\Gamma^h) \cong \Pi_*(\mathcal{N}_\Gamma)/\Pi_*(\mathfrak{A}_\Gamma^h), \quad \mathrm{H}^1(\mathcal{N}_\Gamma/\mathcal{C}_\Gamma) \cong \{0\}.$$

Démonstration (i) Il est immédiat que $\mathfrak{A}_\Gamma^h$ est un idéal de $\mathcal{N}_\Gamma$. De la proposition 3.12, $\Pi_*(\mathfrak{A}_\Gamma^h)$ est une distribution involutive de classe C^∞ de M. Donc en vertu du théorème 3.19, on obtient $[\Pi_*(\mathfrak{A}_\Gamma^h), \Pi_*(\mathfrak{A}_\Gamma^h)] = \Pi_*(\mathfrak{A}_\Gamma^h)$. Par l'isomorphisme de la proposition 3.16, on a $[\mathfrak{A}_\Gamma^h, \mathfrak{A}_\Gamma^h] = \mathfrak{A}_\Gamma^h$. Alors $\mathfrak{A}_\Gamma^h$ est un idéal caractéristique de $\mathfrak{A}_\Gamma$ et de $\mathcal{N}_\Gamma$ d'après une démonstration classique de [4]. Par l'identité de Jacobi, la propriété de $\mathcal{C}_\Gamma$ et le fait que $\mathfrak{A}_\Gamma^h$ est un idéal caractéristique de $\mathcal{N}_\Gamma$, $\mathcal{C}_\Gamma$ est un idéal caractéristique de $\mathcal{N}_\Gamma$.

(ii) Si $\mathfrak{A}_\Gamma^h$ vérifie (H), alors par isomorphisme, $\Pi_*(\mathfrak{A}_\Gamma^h)$ vérifie (H). D'après un théorème de [2], toutes ses dérivations sont des dérivées de Lie par rapport à un champ du normalisateur $\mathcal{N}$. En utilisant les isomorphismes de la proposition 3.16 et de la proposition 3.18, toute dérivation de $\mathfrak{A}_\Gamma^h$ est la dérivée de Lie par rapport à un champ de $\mathcal{N}_\Gamma$. Ainsi, d'après [2], $\mathrm{H}^1\big(\Pi_*(\mathfrak{A}_\Gamma^h)\big) \cong \Pi_*(\mathcal{N}_\Gamma)/\Pi_*(\mathfrak{A}_\Gamma^h)$, et $\mathrm{H}^1(\mathfrak{A}_\Gamma^h) \cong \Pi_*(\mathcal{N}_\Gamma)/\Pi_*(\mathfrak{A}_\Gamma^h)$ en vertu de la proposition 3.16. De même, toute dérivation de $\mathcal{N}$ est intérieure, alors $\mathrm{H}^1(\mathcal{N}) \cong \{0\}$, et $\mathrm{H}^1(\mathcal{N}_\Gamma/\mathcal{C}_\Gamma) \cong \{0\}$ en appliquant la proposition 3.18. ∎

4 Propriétés de la connexion par l'étude de quelques algèbres de Lie qui lui sont associées

Proposition 4.1 *Un champ de vecteurs X de $\mathcal{T}M$ est un élément de $\mathfrak{A}_\Gamma$ si et seulement si X laisse invariant les distributions généralisées définies par les sous-espaces propres de Γ.*

Démonstration Soient $X \in \mathfrak{A}_\Gamma$ et $Y \in h\big(\chi(\mathcal{T}M)\big)$ (resp. $Y \in v\big(\chi(\mathcal{T}M)\big)$). On obtient $\Gamma[X,Y] = [X,\Gamma Y]$, donc $\Gamma[X,Y] = [X,Y]$ (resp. $\Gamma[X,Y] = -[X,Y]$).

Réciproquement, soit $X \in \chi(\mathcal{T}M)$ préservant $h\big(\chi(\mathcal{T}M)\big)$ et $v\big(\chi(\mathcal{T}M)\big)$. Soit Y dans $\chi(\mathcal{T}M)$, Y se décompose en une somme $Y^h + Y^v$ avec $Y^h \in h\big(\chi(\mathcal{T}M)\big)$ et Y^v appartient à $v\big(\chi(\mathcal{T}M)\big)$. On a $\Gamma[X,Y] - [X,\Gamma Y] = ([X,Y^h] - [X,Y^v]) - ([X,Y^h] - [X,Y^v]) = 0$. D'où le résultat. ∎

Proposition 4.2 *Un champ vertical X est un élément de $\mathfrak{A}_\Gamma^v$ si et seulement si X commute avec tout champ horizontal projetable. Si on note par $\mathfrak{A}_{H^0}$ l'algèbre de Lie engendrée par les champs horizontaux projetables, alors $\mathfrak{A}_\Gamma^v$ est son centralisateur dans $\chi(\mathcal{T}M)$.*

Démonstration On sait que tout champ vertical peut s'écrire comme JX avec $X \in h\big(\chi(\mathcal{T}M)\big)$. L'espace vertical est toujours involutif, alors JX est dans $\mathfrak{A}_\Gamma$ si et seulement si JX laisse invariant l'espace horizontal.

Soient U un domaine d'une carte de système de coordonnées (x^i, y^i) de $\mathcal{T}M$, et $X \in h\big(\chi(TU)\big)$. L'ensemble $h\big(\chi(TU)\big)$ étant engendré par $h(\frac{\partial}{\partial x^i})$ sur $F(TU)$. On a $[JX, fh(\frac{\partial}{\partial x^i})] = \big(JX(f)\big)h(\frac{\partial}{\partial x^i}) + f[JX, h(\frac{\partial}{\partial x^i})] \in h\big(\chi(TU)\big)$ pour tout f dans $F(TU)$ est équivalent à $[JX, h(\frac{\partial}{\partial x^i})] = 0$. Comme $\mathfrak{A}_{H^0}$ est un $F(M)$-module, alors son centralisateur dans l'espace horizontal est nul en raisonnant comme dans la proposition 3.14. Donc, son centralisateur est inclus dans l'espace vertical. D'après ce qui précède, $\mathfrak{A}_\Gamma^v$ est le centralisateur de $\mathfrak{A}_{H^0}$. ∎

Théorème 4.3 *Si l'algèbre de Lie $\mathfrak{A}_\Gamma^h$ vérifie (H), alors le premier espace de cohomologie de Chevalley–Eilenberg de $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ est le produit direct de ceux de $\mathfrak{A}_\Gamma^h$ et de $\mathfrak{A}_\Gamma^v$.*

Démonstration D'après la proposition 3.10, $\mathfrak{A}_\Gamma = \mathfrak{A}_\Gamma^h \times \mathfrak{A}_\Gamma^v$ est un produit direct. En vertu du théorème 3.20, $[\mathfrak{A}_\Gamma^h, \mathfrak{A}_\Gamma^h] = \mathfrak{A}_\Gamma^h$; et de la première partie de la proposition 3.14, le centre de $\mathfrak{A}_\Gamma^h$ est nul. Ainsi, d'après le lemme 3.7 de [7, p. 8], on a le résultat. ∎

Théorème 4.4 *Le premier espace de cohomologie de Chevalley–Eilenberg de $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ est nul si et seulement si la connexion Γ est plate.*

Démonstration On suppose que toute dérivation de $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ est intérieure. Or, on a $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R = \mathfrak{A}_\Gamma^h \otimes \mathfrak{A}_\Gamma^v$, alors en vertu de la proposition 4.2, $\mathfrak{A}_\Gamma^h = \mathfrak{A}_{H^0}$. Sur un domaine d'une carte U de coordonnées locales (x^i, y^i), $(\mathfrak{A}_{H^0})_U$ contient chaque élément de base de l'espace horizontal $\frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}$. Comme l'espace vertical est contenu dans $\mathfrak{N}_R$, par suite, chaque $\frac{\partial}{\partial x^i} \in (\mathfrak{N}_R)_U$. Or $(\mathfrak{N}_R)_U$ est une distribution de TU, donc $\chi(U) \subset (\mathfrak{N}_R)_U$, pour tout U. Ainsi, $\mathfrak{N}_R = \chi(\mathcal{T}M)$ et que $R = 0$. C'est-à-dire que la connexion est plate.

Réciproquement, si la connexion est plate, la distribution $\Delta\colon x \in \mathcal{T}M \mapsto \operatorname{Im} h_x$ est intégrable, *cf.* [6]. Pour cette raison, il existe une structure de feuilletage sur $\mathcal{T}M$. Par ailleurs, l'espace de nullité de la courbure $\mathfrak{N}_R$ est $\chi(\mathcal{T}M)$. D'après la proposition 3.12, on a $\mathfrak{A}_\Gamma^h = H^0$. En tenant compte du feuilletage sur $\mathcal{T}M$, on peut affirmer que $\mathfrak{A}_\Gamma^h = \chi(M)$. Et par la proposition 4.2, $\mathfrak{A}_\Gamma^v$ est localement $\chi(\mathbb{R}^n)$. Alors, à l'aide de la proposition 4.3 et d'un résultat de [2], le premier espace de cohomologie de Chevalley–Eilenberg de $\mathfrak{A}_\Gamma \cap \mathfrak{N}_R$ est nul. ∎

Remarque 4.5 Ce théorème est en partie une réponse à une question posée par [6, p. 6] dans le cas où la forme vectorielle est une connexion au sens de Grifone.

Proposition 4.6 *La connexion Γ est plate si et seulement si le centralisateur de $\mathfrak{A}_\Gamma^v$ dans l'espace vertical est réduit à zéro.*

Démonstration Dans le cas où la connexion est plate, $\mathfrak{A}_\Gamma^v$ est localement égal à $\chi(\mathbb{R}^n)$. Ainsi son centralisateur dans l'espace vertical est égal à $\{0\}$ d'après [2]. Réciproquement, par la proposition 4.2, $\mathfrak{A}_{H^0}$ est inclus dans le centralisateur de $\mathfrak{A}_\Gamma^v$. Si le centralisateur vertical de $\mathfrak{A}_\Gamma^v$ est nul, alors $v(\mathfrak{A}_{H^0})$ est réduit à zéro car $\mathfrak{A}_\Gamma^v \subset \mathfrak{A}_\Gamma$. De l'équation (3.1), $R(X, Y) \subset v(\mathfrak{A}_{H^0})$ pour tous $X, Y \in H^0$. On a $R(X, Y) = 0$ pour tous $X, Y \in H^0$. Par le fait que les éléments de H^0 engendrent $h\big(\chi(\mathcal{T}M)\big)$ et que R est linéaire et semi-basique, alors $R(X, Y) = 0$ pour tous $X, Y \in \chi(\mathcal{T}M)$. D'où, la courbure est nulle et la connexion est plate. ∎

Remarque 4.7 La connexion est plate si et seulement si l'algèbre de Lie $\mathfrak{A}_\Gamma^h$ (resp. $\mathfrak{N}_R^h$) est engendrée par tous les champs horizontaux projetables sur $F(M)$ (resp. sur $F(\mathcal{T}M)$).

Exemple 4.8 Soient $M = \mathbb{R}^4$ de système de coordonnées dans $T\mathbb{R}^4$ $(x^i, y^i)_{1 \leq i \leq 4}$, et Γ une connexion telle que $\Gamma_2^1 = -\frac{y^2}{2} e^{x^1}$, $\Gamma_1^2 = \frac{y^2}{2}$, $\Gamma_2^2 = \frac{y^1}{2}$, et les autres nuls. Les coefficients de la courbure R sont

$$R^1_{2,1} = \frac{y^2}{2}e^{x^1}, \quad R^2_{2,1} = -\frac{y^1}{4}$$

et les autres nuls.

Les expressions X^i et Y^i sont dans toute la suite des fonctions C^∞ de $T\mathbb{R}^4$.

L'espace de nullité de la courbure $\mathfrak{N}_R$ est,

$$\mathfrak{N}_R = \left\{ X^3 \frac{\partial}{\partial x^3} + X^4 \frac{\partial}{\partial x^4} + Y^i \frac{\partial}{\partial y^i} \right\}.$$

Ici, l'espace de nullité est involutif car il est engendré par des champs projetables.

On a

$$\mathfrak{N}_R^h = \left\{ X^3 \frac{\partial}{\partial x^3} + X^4 \frac{\partial}{\partial x^4} \right\}.$$

Le centralisateur de $\mathfrak{N}_R^h$ est réduit à zéro, son normalisateur est

$$\mathcal{N}_R = \left\{ X^3 \frac{\partial}{\partial x^3} + X^4 \frac{\partial}{\partial x^4} + \underset{i\neq 3,4}{X^i(\overset{\wedge}{x^3}, \overset{\wedge}{x^4})} \frac{\partial}{\partial x^i} + Y^i(\overset{\wedge}{x^3}, \overset{\wedge}{x^4}) \frac{\partial}{\partial y^i} \right\}$$

où dans toute la suite, $X(\overset{\wedge}{x^i})$ désigne que l'expression de X ne dépend pas de x^i.

Le champ $\frac{\partial}{\partial x^1} - \frac{y^2}{2}\frac{\partial}{\partial y^2}$ est dans la partie horizontale de ce normalisateur mais n'appartenant pas à la partie horizontale de l'espace de nullité. Ce qui confirme que la connexion est non plate.

Dans la suite, $\overline{A}$ désigne la classe d'équivalence de A définie par un quotient d'algèbres de Lie.

On obtient $\mathrm{H}^1(\mathcal{N}_R) \cong \{0\}$. Et $\mathrm{H}^1(\mathfrak{N}_R^h) \cong \mathcal{N}_R/\mathfrak{N}_R^h$ qui est

$$\overline{\left\{ X^i_{(i\neq 3,4)}(\overset{\wedge}{x^3}, \overset{\wedge}{x^4}) \frac{\partial}{\partial x^i} + Y^i(\overset{\wedge}{x^3}, \overset{\wedge}{x^4}) \frac{\partial}{\partial y^i} \right\}}.$$

L'algèbre de Lie $\mathfrak{A}_\Gamma^h$ est donnée par

$$\mathfrak{A}_\Gamma^h = \left\{ X^3(x^t) \frac{\partial}{\partial x^3} + X^4(x^t) \frac{\partial}{\partial x^4} \right\}.$$

Le normalisateur de $\mathfrak{A}_\Gamma^h$ dans $\chi(T\mathbb{R}^4)$ est donc

$$\mathcal{N}_\Gamma = \left\{ X^3(x^t)\frac{\partial}{\partial x^3} + X^4(x^t)\frac{\partial}{\partial x^4} + X^i_{(i\neq 3,4)}(x^t, t\neq 3,4)\frac{\partial}{\partial x^i} + Y^i(x^t, y^j, t\neq 3,4)\frac{\partial}{\partial y^i} \right\}.$$

Et le centralisateur de $\mathfrak{A}_\Gamma^h$ dans $\chi(T\mathbb{R}^4)$ coïncide à

$$\mathcal{C}_\Gamma = \left\{ Y^i(x^t, y^j, t\neq 3,4)\frac{\partial}{\partial y^i} \right\}.$$

Ainsi $\mathcal{N}_\Gamma/\mathcal{C}_\Gamma$ est

$$\overline{\{ X^3(x^t)\frac{\partial}{\partial x^3} + X^4(x^t)\frac{\partial}{\partial x^4} + X^i_{(i\neq 3,4)}(x^t, t\neq 3,4)\frac{\partial}{\partial x^i} \}}.$$

On a alors $\mathrm{H}^1(\mathcal{N}_\Gamma/\mathcal{C}_\Gamma) \cong \{0\}$, et $\mathrm{H}^1(\mathfrak{A}_\Gamma^h) \cong \Pi_*(\mathcal{N}_\Gamma)/\Pi_*(\mathfrak{A}_\Gamma^h)$ est

$$\overline{\{ X^i_{(i\neq 3,4)}(x^t, t\neq 3,4)\frac{\partial}{\partial x^i} \}}.$$

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MR3415660 (Review) 17B66 17B56 53C07

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Sur les algèbres de Lie associées à une connexion. (French. English, French summaries) [Lie algebras associated with a connection]

Canad. Math. Bull. **58** (2015), *no. 4,* 692–703.

This paper studies the first Chevalley-Eilenberg cohomology space of some Lie algebras associated to a Grifone connection (see [J. Grifone, Ann. Inst. Fourier (Grenoble) **22** (1972), no. 1, 287–334; MR0336636] for the definition). Recall that this type of connection is an extension of the direction connection that arises in the Lichnerowicz treatment of Finsler geometry. *Alberto Medina*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

Journal of Generalized Lie Theory and Applications

Randriambololondrantomalala, J Generalized Lie Theory Appl 2015, 9:1
http://dx.doi.org/10.4172/1736-4337.1000217

Research Article Open Access

The *m*-Derivations of Distribution Lie Algebras

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Abstract

Let *M* be an *N*-dimensional smooth differentiable manifold. Here, we are going to analyze (m>1)-derivations of Lie algebras relative to an involutive distribution on subrings of real smooth functions on M. First, we prove that any (m>1)-derivations of a distribution Ω on the ring of real functions on M as well as those of the normalizer of Ω are Lie derivatives with respect to one and only one element of this normalizer, if Ω doesn't vanish everywhere. Next, suppose that *N* = n + q such that n>0, and let *S* be a system of *q* mutually commuting vector fields. The Lie algebra of vector fields $\mathfrak{A}_S$ on M which commutes with *S*, is a distribution over the ring $F_0(\mathrm{M})$ of constant real functions on the leaves generated by *S*. We find that *m*-derivations of $\mathfrak{A}_S$ are local if and only if its derivative ideal coincides with $\mathfrak{A}_S$ itself. Then, we characterize all non local m-derivations of $\mathfrak{A}_S$. We prove that all *m*-derivations of $\mathfrak{A}_S$ and of the normalizer of $\mathfrak{A}_S$ are derivations. We will make these derivations and those of the centralizer of $\mathfrak{A}_S$ more explicit.

AMS Subject Classification: Primary 17B66, 17B40, Secondary 53C12, 53B15, 47B47, 53B40.

Keywords: *m*-derivations; Vector fields Lie algebras; Distributions; Commuting vector fields; Generalized foliations; Compactly supported vector fields; μ-projected vector fields; Nullity space of curvature

Introduction and Preliminary

Let *m* be a natural integer greater than or equal to 2. We recall that an *m*-derivation *D* of a Lie $\mathbb{R}$-algebra $\mathfrak{A}$ is an endomorphism of $\mathfrak{A}$, such that for all $X_1, X_2, \ldots, X_m \in \mathfrak{A}$

$$D\left[X_1,\left[X_2,\ldots,\left[X_{m-1},X_m\right]\ldots\right]\right]=\left[D(X_1),\left[X_2,\ldots\left[X_{m-1},X_m\right]\ldots\right]\right]+$$

$$+\left[X_1,\left[D(X_2),\ldots,\left[X_{m-1},X_m\right]\ldots\right]\right]+\ldots+$$

$$+\left[X_1,\left[X_2,\ldots,\left[D(X_{m-1}),X_m\right]\ldots\right]\right]+$$

$$+\left[X_1,\left[X_2,\ldots,\left[X_{m-1},D(X_m)\right]\ldots\right]\right].$$

This map is inner with respect to Lie algebra if *D* equals to a Lie derivative with respect to $X \in \mathfrak{B}$; if $X \in \mathfrak{A}$, it is an inner *m*-derivation. A standard *m*-derivation *D* is a sum of derivations of $\mathfrak{A}$ and $\mathbb{R}$-linear maps of $\mathfrak{A}$ into the center of $\mathfrak{A}$ such that $D[\mathfrak{A},[\mathfrak{A},[\ldots,[\mathfrak{A},\mathfrak{A}]\ldots]]]=\{0\}$.

Is it sufficient to study derivation of Lie algebras? What is the reason for studying the more general notion:" (*m*>2)-derivation"? In other words, can we find (*m*>2)-derivations of vector fields Lie algebra which are not derivations? In [1], we found *m*-derivations of all polynomial vector fields Lie algebras *P* on $\mathbb{R}^n$, where *P* contains Euler vector fields ***E*** and all constant vector fields. We remark that all these *m*-derivations are derivations when m is even. If *m* is an odd number, *m*-derivations are generally sum of derivations and *m*-derivations with homogeneous degree -2. Over $\mathbb{R}^3$, we can take a simple example where the Lie $\mathbb{R}$-algebra is spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, E, z\frac{\partial}{\partial x}, (z)^2\frac{\partial}{\partial x}$ and the $\mathbb{R}$ linear map *D* is defined by $D\left((z)^2\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial x}$ and vanishing otherwise. It is a 3-derivation, but not a derivation. In [2], some graded Lie algebra *m*-derivations are discussed. Here, we are interested in *m*-derivations of distribution Lie algebra on an *N*-smooth manifold M over an M-real functions ring. We know that all smooth vector fields can be locally approximated to polynomial vector fields, so we think that all results in [1] are naturally true in the case of distributions. But, the results which follow are different. The differential operator theory see [3] is the main tool throughout our proofs.

We denote by *F*(M) the ring of all real functions on M, $\chi(\mathrm{M})$ (resp. $\chi(\mathrm{TM})$) the vector fields Lie algebras over M(resp. over the tangent bundle TM).

At first, we consider an involutive distribution Ω over *F*(M). That is to say, Ω is an *F*(M)-sub-module of the module of all vector fields on M. Assuming that the open set $O_\Omega = \{x \in \mathrm{M} / \Omega(x) \neq \{0\}\}$ equals M, we are looking for characteristics of *m*-derivations of Lie algebras relative to Ω and applications of the obtained results on some remarkable distributions. We propose to prove that each *m*-derivation of Ω (resp. of the normalizer in $\chi(\mathrm{M})$ of Ω) is simply a Lie derivative with respect to one and only one normalizer's vector fields (resp. is inner). These theorems can be extended where O_Ω is dense over M.

Secondly, let be *N*=*n*+*q* with $n \geq 1$ and *q*>0, *S* a system of *q* non-vanishing vector fields which commute mutually. We know by results in [4] that *S* yields a generalized foliation on M. We assume that all leaves are regular and we notice that $F_0(\mathrm{M})$, the ring of real smooth functions which are constant on the leaves over M. Let *U* be a *p*-dimensional adapted chart domain relative to the foliation and (U, x^a, y^i) (resp. (U, x^a)), where $1 \leq a \leq n+q-p, 1 \leq i \leq p$ if $p \geq 1$ (resp. where $1 \leq a \leq n+q$ if p=0). Then, there are two modules over $F_0(U)$, $A_S^1(U)$ spanned by $\left(\frac{\partial}{\partial x^a}\right)_{1\leq a\leq n+q-p}$ and $\mathfrak{A}_S^2(U)$ generated by $\left(\frac{\partial}{\partial y^i}\right)_{1\leq i\leq p}$. These previous modules are Lie algebras such that $\mathfrak{A}_S(U)$ is equal to the semi-

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Received November 14, 2014; **Accepted** March 07, 2015; **Published** March 16, 2015

Citation: Randriambololondrantomalala P (2015) The *m*-Derivations of Distribution Lie Algebras. J Generalized Lie Theory Appl 9: 217. doi: 10.4172/1736-4337.1000217

direct product of these two algebras: $\mathfrak{A}_S(U)=\mathfrak{A}_S^1(U)\oplus\mathfrak{A}_S^2(U)$ for all distinguished *U*. We can say that $\mathfrak{A}_S$ is a smooth distribution of M over $F_0(\mathrm{M})$. Throughout this paper, we assume that this chart is $(p>0)$-dimensional in the sense of the foliation, unless expressly stated. Our aims are to characterize all m-derivations of $\mathfrak{A}_S$, of the normalizer of $\mathfrak{A}_S$ and of the centralizer of $\mathfrak{A}_S$ in $\chi(\mathrm{M})$. The corresponding work where $S=\{0\}$ has been done in the previous section. Because of $\mathfrak{A}_S$'s lower central series constancy, which coincides with the module direct sum of $\mathfrak{A}_S^1\oplus\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]$, our work on these *m*-derivations is non-trivial. The main results of this section are: every *m*-derivation is local iff the derivative ideal of $\mathfrak{A}_S$ is $\mathfrak{A}_S$ itself, which is equivalent to the fact that $\mathfrak{A}_S^1(F_0(\mathrm{M}))$ has non-vanishing element over the whole M. Moreover, all *m*-derivations of $\mathfrak{A}_S$ or of the normalizer $\mathfrak{N}$ of $\mathfrak{A}_S$ are sums of a Lie derivative with respect to one $\mathfrak{N}$'s element, of one local *m*-derivation which takes its value in $\mathfrak{A}_S^2$ depending on two non-vanishing 1-differential forms over $\mathfrak{A}_S$, and of a non-local m-derivation of $\mathfrak{A}_S$. We give some recommendations for constructing all these non local *m*-derivations. In addition, all $\mathbb{R}$-linear maps of $\langle S\rangle$ the centralizer of $\mathfrak{A}_S$ into itself are *m*-derivations. We characterize all local $\mathbb{R}$-endomorphisms of $\langle S\rangle$ in the case where all elements of *S* are densely supported or $\langle S\rangle$ is spanned by singleton, and those which are non local. It is well known that the open set of all foliation regular points is dense in M, then one can extend these results where the foliation is singular and if the above 1-forms prolong smoothly on M.

Several applications of our results about Lie algebras relative to: all vector fields, all compactly supported vector fields, generalized foliations, μ-projected vector fields cf. [5], *k*-nullity space of connection curvature, and vector fields Lie algebras on TM commuting with Liouville vector fields cf. [6]; are given at the end of this paper.

Throughout this article, the Lie derivative with respect to $X\in\chi(\mathrm{M})$ is denoted $\boldsymbol{L}_x$. We adopt the Einstein index summation and suppose that all considered objects are smooth.

The *m*-derivations of Lie algebras attached to Ω

According the hypothesis about Ω, we can affirm that Ω is a Lie sub-algebra of $\chi(\mathrm{M})$. A generalization of [7,8]'s theorems in the sense of derivation or triple derivation can be stated as follows:

Theorem 2.1. *All m-derivations of Ω (resp. of the normalizer of Ω in* $\chi(\mathrm{M})$*) are Lie derivative with respect to one and only one vector field of the normalizer of Ω (resp. is inner).*

Proof. Assume that $x\in\mathrm{M}$, $\exists X\in\Omega$ such that $X(x)\neq 0$. By Frobenius theorem, we find one chart (U_x,φ_x) which contains x and local coordinate system $(x^1,\dots,x^{n-1},y)$ where $X\underset{U_x}{=}\frac{\partial}{\partial y}$. Letting *D* be an *m*-derivation of Ω, we know that the Lie algebra spanned by brackets of all elements in Ω is the derivative ideal of Ω denoted by [Ω; Ω]. Local behavior of *D* can be proved by adapting one of Proposition 2.4 in [7] and using that the derivative ideal of Ω is Ω itself. Therefore D_{U_x} is an *m*-derivation of Ω_{U_x}. Let's give $f\in F(U_x)$, as we know, $\frac{\partial}{\partial y}\in\Omega_{U_x}$ then $D_{U_x}\left(f\frac{\partial}{\partial y}\right)=D^0(f)\frac{\partial}{\partial y}+\underset{0<a<n}{D^a(f)}\frac{\partial}{\partial x^a}$ is uniquely determined, where each D^i is differential operator over the trivial bundle $U_x\times\mathbb{R}$ cf. [3]. Thus, if necessary we can write $D^0=\sum_{|A|\geq 0,|B|\geq 1,q\geq 1,r\geq 1}\chi^{A,0}\frac{\partial^{|A|}}{\partial x^A}+\chi^{B,q}\frac{\partial^{|B|+q}}{\partial x^B\partial y^q}+\chi^{0,r}\frac{\partial^r}{\partial y^r}$, where *A*, *B* are multi-indices corresponding to coordinates.

Let's apply D_{U_x} to $\left[x^j\frac{\partial}{\partial y},\left[y\frac{\partial}{\partial y},\dots,\left[y\frac{\partial}{\partial y},f\frac{\partial}{\partial y}\right]\dots\right]\right]$, where $x^n=y$. By definition of *m*-derivations and when *f* is replaced by monomials, we have:

- If $deg(f)\geq 2$, $D^0(f)(0,\dots,0)=0$ except for $f\equiv y^2$.
- If $deg(f)\geq 1$, $D^a(f)(0,\dots,0)=0$ except for $f\equiv yP(x^i,i\neq n)$

where $P(x^i)$ are free x^n monomials.

By reasoning as in the previous, we compute $D_{U_x}\left[\frac{\partial}{\partial y},\left[\frac{\partial}{\partial y},\dots,\left[\frac{\partial}{\partial y},y^mP(x^i,i\neq n)\frac{\partial}{\partial y}\right]\dots\right]\right]$. It is easy, using both the previous relation and the previous proof, to obtain the nullity of $D^a\left(yP(x^i)\right)(0,\dots,0)$. By coordinates translations, we can affirm that each D^j, $j\neq 0$ is a differential operator of order 0 and D^0 is a sum of one of order 1 with one other $\chi^{0,2}\frac{\partial^2}{(\partial y)^2}$ of order 2.

Computing in the same way as the previous calculus, $D_{U_x}\left[f\frac{\partial}{\partial y},\left[y\frac{\partial}{\partial y},\dots,\left[y\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right]\dots\right]\right]$ gives:

- for $f\equiv y$, $D^0(1)=-\frac{\partial\chi^{0,1}}{\partial y}$.
- for $f\equiv x^j$ except $j\neq n$, $D^j(1)=-\frac{\partial\chi^{(0,\dots,0,1_j,0,\dots,0)}}{\partial y}$ (1_j means 1 is in *j*-th rank).

By these results, $D_{U_x}\left(f\frac{\partial}{\partial y}\right)=\mathrm{L}_{\chi^{0,1}\frac{\partial}{\partial y}+\chi^{(0,\dots,0,1_a,0,\dots,0)}\frac{\partial}{\partial x^a}}\left(f\frac{\partial}{\partial y}\right)+\chi^{0,2}\frac{\partial^2 f}{(\partial y)^2}\frac{\partial}{\partial y}$. Consequently,

$D_0=\chi^{0,2}\frac{\partial^2}{(\partial y)^2}\otimes\frac{\partial}{\partial y}$ is an *m*-derivation of the $F(U_x)$-sub-module spanned by $\frac{\partial}{\partial y}$. Applying D_0 to $\left[y^2\frac{\partial}{\partial y},\left[\frac{\partial}{\partial y},\dots,\left[\frac{\partial}{\partial y},y^{m-1}\frac{\partial}{\partial y}\right]\dots\right]\right]$, we have $\chi^{0,2}(y=0)=0$. By coordinate's translations, we can write that $\chi^{0,2}=0$.

We take Proposition 2.6 of [7] and we have $D_{U_x}=\mathrm{L}_{\chi^{0,1}\frac{\partial}{\partial y}+\chi^{(0,\dots,0,1_a,0,\dots,0)}\frac{\partial}{\partial x^a}}$.

Following the arguments of the proof of Theorem 2.7 in [7], we end the demonstration of the first assertion of our theorem. Taking that the derivative ideal of Ω is Ω itself into account, we can adapt the proof of Theorem 2.12 in [7] to state the second assertion.

Remark 2.2. These theorems are correct if we consider O_Ω to be dense over M and if the corresponding vector of the Lie derivative relative to the *m*-derivation cited by Theorem 2.1 can be smoothly extended towards M.

The *m*-derivations of Lie algebras defined by $\mathfrak{A}_S$

We know that nilpotency of order $m-1$ of $\mathfrak{A}_S$ forces any endomorphism of $\mathfrak{A}_S$ to be an *m*-derivation. To avoid this triviality, we prove that:

Proposition 3.1. *The lower central series of* $\mathfrak{A}_S$ *are constant and equal to the module* $\mathfrak{A}_S^1\oplus\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]$.

Proof. The lower central series of is determined by $\mathfrak{C}^0(\mathfrak{A}_S)=\mathfrak{A}_S$ and for all $p>0$,

$\mathfrak{C}^p(\mathfrak{A}_S)=\left[\mathfrak{A}_S,\mathfrak{C}^{p-1}(\mathfrak{A}_S)\right]$ cf.[1]. By Proposition 3.7 of [4], the derivative ideal of $\mathfrak{A}_S$ is $\mathfrak{A}_S^1\oplus\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]$. From the linearity of brackets, the Jacobi identity and the fact that $\mathfrak{A}_S^2$ is an ideal of $\mathfrak{A}_S$, we have $\left[\mathfrak{A}_S,\left[\mathfrak{A}_S^1\oplus\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]\right]\right]=\mathfrak{A}_S^1\oplus\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]$. Then, we deduce the result.

We assume the following conventions about the indices, $i,j,k\in\{1,\dots,p\}$, $a,b,c\in\{1,\dots,n+q-p\}$, and each index indexed by 0 is fixed.

Proposition 3.2. *Let D be an m-derivation of* $\mathfrak{A}_S$ *and U a domain of distinguished chart such that if* $X\in\mathfrak{A}_S$ *over U vanishes, then* $D(X)$ *over U on* $\mathfrak{A}_S^1$ *is zero.*

Proof. Let D be a such m-derivation and X an element of $\mathfrak{A}_S$ satisfying the above hypothesis. We assume that $D(X)_{|A_S^1}\underset{U}{\neq}0$, then it exists an open set V_z containing z, such that the a_0-th component of $D(X)_{|A_S^1}$ on V_z is everywhere non zero. Let's consider $f\in F_0(\mathrm{M})$ such that $f_{|V_z}=\left(x^{a_0}\right)^2$ where $\mathrm{Supp}(f)\subset U$, and $Y,X_3,\ldots,X_m$ are elements of $\mathfrak{A}_S$ with $Y_{|V_z}=\frac{\partial}{\partial y^{i_0}}$, $X_{3|V_z}=\ldots=X_{m-1|V_z}=x^{a_0}\frac{\partial}{\partial x^{a_0}}$, $X_{m|V_z}=\frac{\partial}{\partial x^{a_0}}$.
By definition, we obtain

$$D\left[X,\left[fY,\left[X_3,\ldots\left[X_{m-1},X_m\right]\ldots\right]\right]\right](z)=\left[D(X),\left[fY,\left[X_3,\ldots\left[X_{m-1},X_m\right]\ldots\right]\right]\right](z)+[X,Z](z) \qquad (3.1)$$

with $Z\in\mathfrak{A}_S$, a contradiction.

Proposition 3.3. *The centralizer* $\mathcal{C}$ *of* $\mathfrak{A}_S^1\oplus\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]$ *coincides with the vector* $\mathbb{R}$ *-space* $\langle S\rangle$ *spanned by S.*

Proof. Recall that

$$\mathcal{C}=\left\{X\in\chi(\mathrm{M})/\left[X,\mathfrak{A}_S^1\oplus\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]\right]=\{0\}\right\}.$$

Choose $X\in\mathcal{C}$ and let be U a distinguished connected chart domain of the foliation. When $p=0$, we have $X=0$. For $p\geq 1$, we put $X_{|U}=X^a\frac{\partial}{\partial x^a}+X'^i\frac{\partial}{\partial y^i}\in\mathfrak{C}_U$. By the fact $\left[X,\mathfrak{A}_S^1\right]=\{0\}$, $X^a=0$ for all a and each $X'^i\in\left(F(U)-F_0(U)\right)\cup\left(F(U)\cap F_0(U)\right)$. Therefore, $X_{|\mathfrak{A}_S^1}=0$ and $X=f^iX_i$ where all $f^i\in\left(F(\mathrm{M})-F_0(\mathrm{M})\right)\cup\left(F(\mathrm{M})\cap F_0(\mathrm{M})\right)$. Assume $Y^1\left(g^i\right)X_i\in\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]$ with $g^jX_j\in\mathfrak{A}_S^2, Y^1\in\mathfrak{A}_S^1$. It's known that $\left[X,\mathfrak{A}_S^1\oplus\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]\right]=\{0\}$ so $Y^1\left(g^i\right)X_i\left(f^j\right)X_j=0$ for all g^i and Y^1. Then all f^j are in $F_0(\mathrm{M})$ and consequently they are constant, and $\mathcal{C}$ is a subset of the $\mathbb{R}$ -vector space spanned by *S*. The converse inclusion is obvious.

Proposition 3.4. *All non-local m-derivations of* $\mathfrak{A}_S$ *vanish on* $\mathfrak{A}_S^1\oplus\left[\mathfrak{A}_S^1,\mathfrak{A}_S^2\right]$ *and take their values in* $\langle S\rangle$*. Conversely, all* $\mathbb{R}$ *-endomorphisms D of* $\mathfrak{A}_S$ *which have these properties, is an m-derivation of* $\mathfrak{A}_S$*. All theses maps are standard m-derivations.*

Proof. To simplify, we pose a such m-derivation D. Then there is $X\in\mathfrak{A}_S$ and a distinguished chart domain U so that $X_{|U}\equiv 0$ with $D(X)(z\in U)\neq 0$. Thus, we have an open set V_z in U containing z, with $D(X)$ everywhere non-vanishing. Recall that the center of $\mathfrak{A}_S$ is the intersection of its centralizer with itself. We reason by contradiction, we suppose that $D(X)$ doesn't belong to the center of $\mathfrak{A}_S$. By Proposition 3.2, we claim that on V_z, the i_0-th component of $D(X)$ is everywhere non vanishing. So, this component is not a constant function. Consequently, we can assume that its partial derivative with respect to a x^{a_0} is non-zero at *z*. Then, we consider $X_2,\ldots,X_m$ to be elements of $\mathfrak{A}_S$ such that $\mathrm{Supp}(X_2)\subset U$ and $X_{2|V_z}=\ldots=X_{m-1|V_z}=x^{a_0}\frac{\partial}{\partial x^{a_0}}$, $X_{m|V_z}=\frac{\partial}{\partial x^{a_0}}$. By the m-derivation definition,

$$D\left[X,\left[X_2,\left[X_3,\ldots\left[X_{m-1},X_m\right]\ldots\right]\right]\right](z)=\left[D(X),\left[X_2,\left[X_3,\ldots\left[X_{m-1},X_m\right]\ldots\right]\right]\right](z)+[X,Z](z)$$

where $Z\in\mathfrak{A}_S$, we have a contradiction. Moreover, Proposition 3.1 and the previous result lead to nullity of *D* over $\mathfrak{A}_S^1\oplus\left[A_S^1,\mathfrak{A}_S^2\right]$.

It is easy to prove the last assertions of our proposition.

We can note immediately that,

Lemma 3.5. *For all* $k\geq 2$*, if D is a k-derivation of Lie a algebra* $\mathfrak{A}$ *then the center* C *of* $\mathfrak{A}$ *satisfies the following equation* $\left[D(\mathrm{C}),\mathfrak{C}^{k-2}(\mathfrak{A})\right]=\{0\}$.

Proposition 3.6. *Local m-derivations of* $\mathfrak{A}_S$ *stabilize* $\mathfrak{A}_S^2$.

Proof. We set a local m-derivation D, $D_{|U}$ is still an m-derivation. Without trivial case $p=0$, let a,b,i be some fixed indices, we write

$$D_U\left(x^b\frac{\partial}{\partial y^i}\right)=D_{i,b}^c\frac{\partial}{\partial x^c}+D_{i,b}^{j+m+q-p}\frac{\partial}{\partial y^j}$$

and

$$(-1)^{m-1}\delta_a^bD_U\left(\frac{\partial}{\partial y^i}\right)=D_U\left(\left[x^b\frac{\partial}{\partial y^i},\left[x^a\frac{\partial}{\partial x^a},\left[\ldots,\left[x^a\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^a}\right]\ldots\right]\right]\right]\right),\forall a. \qquad (3.2)$$

By using the equality (3.2), Lemma 3.5 and Proposition 3.3, we deduce that each $D_{i,b}^c$ is constant.

Let *f* be an element of $F_0(U)$, we remark that

$$\left[f\frac{\partial}{\partial x^b},\left[x^c\frac{\partial}{\partial x^c},\left[\ldots,\left[x^c\frac{\partial}{\partial x^c},x^b\frac{\partial}{\partial y^i}\right]\ldots\right]\right]\right]=f\frac{\partial}{\partial y^i}. \qquad (3.3)$$

By mapping D_U to (3.3) in the case where *f* is a polynomial of degree greater or equal than two, the previous result and the fact that $D_{i,b}^c$ is a differential operator over $U\times\mathbb{R}$, prove that $D_U\left(f\frac{\partial}{\partial y^i}\right)\in\mathfrak{A}_S^2(U)$ for all i. Furthermore, combining the previous results and the obtained relation by

$$D_U\left[\left(x^b\right)^2\frac{\partial}{\partial x^b},\left[x^c\frac{\partial}{\partial x^c},\left[\ldots,\left[x^c\frac{\partial}{\partial x^c},\frac{\partial}{\partial x^b}\right]\ldots\right]\right]\right]=-2D_U\left(x^b\frac{\partial}{\partial y^i}\right)$$

we see that $D_{i,b}^c=0$. Then, $D_U\left(x^b\frac{\partial}{\partial y^i}\right)\in\mathfrak{A}_S^2(U)$.

In addition,

$$D_U\left(f\frac{\partial}{\partial y^i}\right)=D_U\left[f\frac{\partial}{\partial x^b},\left[x^c\frac{\partial}{\partial x^c},\left[\ldots,\left[x^c\frac{\partial}{\partial x^c},x^b\frac{\partial}{\partial y^i}\right]\ldots\right]\right]\right]$$

and the previous statement lead to $D_U\left(f\frac{\partial}{\partial y^i}\right)\in\mathfrak{A}_S^2(U)$ for all *U*.

Proposition 3.7. *The Lie algebra* $\mathfrak{A}_S^2$ *is stabilized by m-derivations of* $\mathfrak{A}_S$.

Proof. We deduce the result from Propositions 3.4, 3.6.

Theorem 3.8. *We have equivalences between:*

1. *All m-derivations of* $\mathfrak{A}_S$ *are local.*
2. *There is an* $X\in A_S^1$ *and* $h\in F_0(\mathrm{M})$ *such that* $X(h)(x)\neq 0\ \forall x\in\mathrm{M}$.
3. *The derivative ideal of* $\mathfrak{A}_S$, $\left[\mathfrak{A}_S,\mathfrak{A}_S\right]$ *coincides with* $\mathfrak{A}_S$ *itself.*

Proof. In $2.\Rightarrow 1.$, we use the same reasoning as the one of the proof of Theorem 3.11 in [4]. As for $1.\Rightarrow 2.$ we suppose that there is an $f\notin F_0(\mathrm{M})\mathfrak{A}_S^1(F_0(\mathrm{M}))$. Since $S\neq 0$, then it exists k such that X_k is non-zero on the open set U_k, and $g\in F_0(\mathrm{M})$ vanishing on U_k with $(fg)_{|U_k}\equiv 0$. So, it is immediate that the $\mathbb{R}$ -linear map defined by

$$D(X)=\begin{cases}0 & \text{if } X\in\mathfrak{A}_S\text{-}\{\mathbb{R}\,\mathrm{fg}\,X_k\},\\ D^jX_j & \text{if } X=\mathrm{fg}\,X_k \text{ where } D^j\in\mathbb{R} \text{ forall j=1,}\ldots,\text{q.}\end{cases}$$

Citation: Randriambololondrantomalala P (2015) The *m*-Derivations of Distribution Lie Algebras. J Generalized Lie Theory Appl 9: 217. doi: 10.4172/1736-4337.1000217

is a non-local m-derivation when $D^k \neq 0$. Thus $1. \Leftrightarrow 2.$.

We reason in the same way as in [4] for $1. \Leftrightarrow 3.$.

Remark 3.9. We assert that if the derivative ideal of $\mathfrak{A}_S$ doesn't coincide with $\mathfrak{A}_S$, then it exists $f \notin F_0(\mathrm{M})\mathfrak{A}_S^1(F_0(\mathrm{M}))$, zero on the open set where one X_k is non-vanishing. To realize a non-local m-derivation D, we exploit the non-vanishing on $\mathfrak{A}_S - \left(\mathfrak{A}_S^1 \oplus \left[\mathfrak{A}_S^1, \mathfrak{A}_S^2\right]\right)$ of the following $\mathbb{R}$-linear map:

For $X = gX_i$, $D(X) = D_{g,i}^j X_j$ where

$$\begin{cases} D_{f,k}^k \neq 0 \textit{ or} \\ \text{if there is a non-vanishing } X_{i_0} \text{ over the previous cited open set and } D_{f,k}^{i_0} \neq 0 \textit{ or} \\ \exists i_0, j_0; h \notin F_0(\mathrm{M})\mathfrak{A}_S^1(F_0(\mathrm{M})) / \mathrm{Supp}(X_{j_0}) \neq \mathrm{M} \text{ and } \complement \mathrm{Supp}(X_{j_0}) \subset \mathrm{Supp}(X_{i_0}) \exists D_{h,j_0}^{i_0} \neq 0 \\ \textit{or it exists } X_{k'} \textit{ and } h \notin F_0(\mathrm{M})\mathfrak{A}_S^1(F_0(\mathrm{M})) \text{ such that their multiplication is zero on an} \\ \text{open set } V \text{ without non-vanishing both over } V \text{ and if there is } X_{k_0} \text{ on } V \; D_{h,k'}^{k_0} \neq 0 \end{cases}$$

These results are immediate by using Theorem 3.8, Proposition 3.4 and the definition of non-local $\mathfrak{A}_S$ m-derivations.

Proposition 3.10. *The normalizer $\mathfrak{N}$ of $\mathfrak{A}_S$ in ( ) is locally isomorphic to $\mathfrak{A}_S \oplus gl(p, \mathbb{R})$ as a vector space, where p is the corresponding leaf local dimension. So $[\mathfrak{N}, \mathfrak{A}_S] = \mathfrak{A}_S$, locally $\left[\mathfrak{N}, \mathfrak{A}_S^1 \oplus \left[\mathfrak{A}_S^1, \mathfrak{A}_S^2\right]\right] = \mathfrak{A}_S$. Moreover, all local m-derivations of $\mathfrak{N}$ stabilize $\mathfrak{A}_S$.*

Proof. We define $\mathfrak{N}$ by the set of all vector fields X such that $[X, \mathfrak{A}_S] \subset \mathfrak{A}_S$. So, we are in a distinguished chart U, all $\mathfrak{N}_{|U}$'s elements are obtained with direct use of the definition of the normalizer of $\mathfrak{A}_S(U)$. Indeed, $\mathfrak{N}_{|U}$ is the sum of $\mathfrak{A}_S(U)$ and the vector $\mathbb{R}$-space spanned by $y^i \dfrac{\partial}{\partial y^j}$. It's clear that, this last space is isomorphic to $gl(p, \mathbb{R})$. The two results which follow are easily proved by the same argument as the previous. As for the last assertion, let's take $X_1 \in \mathfrak{N}$, $X_2, \ldots, X_m \in \mathfrak{A}_S$ and D a local m-derivation of $\mathfrak{N}$. In accordance with the m-derivation definition, we have $D[X_1, [X_2, \ldots, [X_{m-1}, X_m] \ldots]] \in \mathfrak{A}_S$. By local equation $\left[\mathfrak{N}, \mathfrak{A}_S^1 \oplus \left[\mathfrak{A}_S^1, \mathfrak{A}_S^2\right]\right] = \mathfrak{A}_S$, Proposition 3.1 and the previous result where each X_i runs through over the respective sets, we affirm that $D_{|\mathfrak{A}_S}(\mathfrak{A}_S)$ is a subset of $\mathfrak{A}_S$.

Theorem 3.11. *Given that we have a local m-derivation D of $\mathfrak{A}_S$ towards $\mathfrak{A}_S^2$. We find 1-differential closed forms α^i and ω^i over U, where $i = 1, \ldots, p$ with $D_U = (\alpha^j + \omega^j) \otimes \dfrac{\partial}{\partial y^j}$ denoted $D_U^{(\alpha,\omega)}$ such that $ker(\alpha^j) \supset \mathfrak{A}_S^2(U)$ and $ker(\omega^j) \supset \mathfrak{A}_S^1(U)$. Besides,*

$$\alpha^i[X_1, X_2] = X_1.\alpha^i(X_2) - X_2.\alpha^i(X_1)$$

$\forall X_1, X_2 \in \mathfrak{A}_S(U)$. The converse of this result is also true. Furthermore, the condition that the maps $D_U^{(\alpha,\omega)}$, with $\alpha = (\alpha^1, \ldots, \alpha^p)$ and $\omega = (\omega^1, \ldots, \omega^p)$, are inner is equivalent to, for all i, $\omega^i \equiv 0$ and α^i are exact. Then we get $D_U^{(\alpha,\omega)} = -\mathrm{L}_{f^i \frac{\partial}{\partial y^i}}$ where, $\alpha^i = df^i$ with $f^i \in F_0(U)$. Generally if $\alpha = 0$, then $D_U^{(\alpha,\omega)} = \mathrm{L}_{y^i C_i^j \frac{\partial}{\partial y^j}}$ where $C_i^j \in \mathbb{R}$.

Proof. Agreeing with the above hypothesis, we pose $D_U = \gamma^i \otimes \dfrac{\partial}{\partial y^i}$, where the $\gamma^i = \gamma_a^i dx^a + \gamma_j'^i dy^j$ with γ_a^i, $\gamma_j'^i$ belong to $F_0(\mathrm{U})$. By the relations of m-derivations which come from

$$D_U\left[\frac{\partial}{\partial y^{j_0}}, \left[x^a \frac{\partial}{\partial x^a}, \ldots, \left[x^a \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^{a_0}}\right] \ldots\right]\right],$$

we state that $\dfrac{\partial \gamma_{j_0}'^i}{\partial x^{a_0}} = 0$ for all j_0 and a_0. We write the subsequent equality

$$D_U\left[\frac{\partial}{\partial x^{a_0}}, \left[x^a \frac{\partial}{\partial x^a}, \ldots, \left[x^a \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^{b_0}}\right] \ldots\right]\right] \text{ at } (0, \ldots, 0).$$

Then, we can have $\left(\dfrac{\partial \gamma_{a_0}^i}{\partial x^{b_0}} - \dfrac{\partial \gamma_{b_0}^i}{\partial x^{a_0}}\right)(0, \ldots, 0) = 0$, for all a_0, b_0. So, with the help of coordinate's translations, we get the previous equality at other arbitrary points in U. Thus, each γ^i is closed. By exploiting all these assertions, we can adapt the demonstrations of Proposition 3.14, 3.15 et 3.16 of [4] and we achieve our proof.

Let $\mathfrak{R}$ be the set of pair of forms (α, ω) quoted before. We will denote by $\mathfrak{Z}(U)$, the complement set of those of $(\alpha, \omega) \in \mathfrak{R}$ such that α is exact and $\omega = 0$ or $\alpha = 0$. We might assume that $\mathfrak{Z} = \mathfrak{Z}(\mathrm{M})$.

Theorem 3.12. *The form of m-derivations of $\mathfrak{A}_S$ is $\mathrm{L}_X + D^{(\alpha,\omega)\in\mathfrak{Z}} + D_1$ where $X \in \mathfrak{N}$, for every distinguished chart U, $D^{(\alpha,\omega)}{}_{|U} = 0$ if the leaf dimension over U is zero; $D^{(\alpha,\omega)}{}_{|U} = D_U^{(\alpha,\omega)\in\mathfrak{Z}_U}$ otherwise. And D_1 is a non-local m-derivation analogous to the one of Remark 3.9. Particularly, these m-derivations are derivations.*

Proof. Taking D an m-derivation of $\mathfrak{A}_S$, it is split into a sum of local m-derivation D_0 and of a non-local m-derivation D_1 of $\mathfrak{A}_S$. So, D_1 has the same form as the one of Remark 3.9. We can write D_0 as $D_0^{11} + D_0^{12} + D_0^{21} + D_0^{22}$ with D_0^{lt} the $\mathbb{R}$-linear component of D_0 mapping $\mathfrak{A}_S^l$ to $\mathfrak{A}_S^t$, where $l, t = 1, 2$. By Proposition 3.7, $D_0^{21} = 0$. In accordance with the same proposition, we can divide $\mathfrak{A}_S$ by $\mathfrak{A}_S^2$ and the quotient m derivation of D_0 is denoted $\overline{D_0}$. The map $\overline{D_0}$ becomes an m-derivation of $\overline{\mathfrak{A}_S^1}$ by the splitting of D_0. We know that $\mathfrak{A}_S^l$ is locally isomorphic to $\chi\left(\mathbb{R}^{n+q-p}\right)$. Then $\overline{D_0}$ coincides with L_X where $X \in \overline{\mathfrak{A}_S^1}$. Consequently, $D_0 = \mathrm{L}_Y + D^{(\alpha,\omega)}$ with $Y \in \mathfrak{N}$ and $D^{(\alpha,\omega)}$, $(\alpha, \omega) \in \mathfrak{Z}$ is defined by Theorem 3.11. With the help of Proposition 3.4, respectively Theorem 3.11, D_1 respectively D_0 is a derivation.

Proposition 3.13. *Each m-derivation of the Lie $\mathbb{R}$-algebra of all linear fields taking value in the constant fields Lie $\mathbb{R}$-algebra of $\mathbb{R}^t$ is Lie derivative with respect to one constant field.*

Proof. Let D be such m-derivation and $(z^l)_{1 \le l \le t}$ one coordinates system of $\mathbb{R}^t$. We note that $D\left(z^u \dfrac{\partial}{\partial z^v}\right) = D_v^{ul} \dfrac{\partial}{\partial z^l}$ with $u, v \in \{1, \ldots, t\}$. It's easy to verify that D vanishes if and only if $D(E)$ is zero too by using the following equation

$$D\left[E, \left[E, \ldots, \left[E, z^u \frac{\partial}{\partial z^v}\right] \ldots\right]\right] = \left[E, \left[E, \ldots, \left[E, D\left(z^u \frac{\partial}{\partial z^v}\right)\right] \ldots\right]\right] \quad (3.4)$$

where E is the Euler vector field. Then, we write $D(E) = C^l \dfrac{\partial}{\partial z^l}$. For different and fixed u, v, we exploit the obtained relation from

$$D\left[z^u \frac{\partial}{\partial z^u}, \left[z^u \frac{\partial}{\partial z^u}, \ldots, \left[z^u \frac{\partial}{\partial z^u}, z^u \frac{\partial}{\partial z^v}\right] \ldots\right]\right].$$

Therefore, we have $D_v^{ul} = 0$ where $l \neq u, v$; $D_v^{uv} = D_u^{uu}$. In accordance with (3.4) when u=v, we state that $C^u = D_u^{uu}$ and $D_u^{ul} = 0$ for $l \neq u$. In addition, (3.4) gives us $D_v^{uu} = 0$. Thus, we proved that $D = \mathrm{L}_{C^l \frac{\partial}{\partial z^l}}$.

Proposition 3.14. *Every m-derivation of $\mathfrak{N}$ taking its value in*

$\langle S\rangle$ *is a sum of m-derivations of* $\mathfrak{A}_S$ *towards* $\langle S\rangle$ *and m-derivations of* $(\mathfrak{N}-\mathfrak{A}_S)\cup\{0\}$ *to* $\langle S\rangle$.

Proof. Let D be an m-derivation of $\mathfrak{N}$ towards $\langle S\rangle$. It is known that every local m-derivation of $\mathfrak{N}$ stabilizes $\mathfrak{A}_S$, $(\mathfrak{N}-\mathfrak{A}_S)\cup\{0\}$ is a Lie algebra and there is a direct sum of modules $\mathfrak{N}=\mathfrak{A}_S\oplus((\mathfrak{N}-\mathfrak{A}_S)\cup\{0\})$ Moreover, every non-local m-derivation of $\mathfrak{N}$ vanishes on $\mathfrak{A}_S\subset[\mathfrak{N},\mathfrak{N}]$. Then $D=D^{11}+D^{21}$ with D^{11} (resp. D^{21}) the linear component map of D from $\mathfrak{A}_S$ to $\langle S\rangle$ (resp. of $(\mathfrak{N}-\mathfrak{A}_S)\cup\{0\}$ towards $\langle S\rangle$). By the fact that D is linear and D^{11} takes value in $\langle S\rangle$; D^{11}, D^{21} are m-derivations.

Proposition 3.15. *All m-derivations from* $(\mathfrak{N}-\mathfrak{A}_S)\cup\{0\}$ *towards* $\langle S\rangle$ *are Lie derivative with respect to an element of* $\langle S\rangle$.

Proof. For $k\in\{1,\ldots,q\}$, we consider the open set $U_k=\{x\in M / X_k(x)\neq 0\}$. We know that all elements of $(\mathfrak{N}-\mathfrak{A}_S)\cup\{0\}$ are of the form Cf^iX_j where $C\in\mathbb{R}$, $f^i\in F_0(M)$ such that $X_i(f^j)$ in U_i equals 0 for all $j\neq i$, equals 1 for $j=i$. First, we show that such an m-derivation D is local. We fix i,j belonging to $\{1,\ldots,q\}$ with $i\neq j$. Only in $\complement\mathrm{Supp}(X_i)$ we can find an distinguished open set U such that $(f^jX_i)_{|U}\equiv 0$ or $(f^iX_i)_{|U}\equiv 0$. It is immediate that $[f^iX_i,[f^iX_i,\ldots,[f^iX_i,f^jX_i]\ldots]]=(-1)^{m-1}f^jX_i$. By applying D to this last expression, we obtain $(D(f^jX_i))_{|U}\equiv 0$. Now, we let the map D acts on the following bracket $[f^jX_j,[f^jX_j,\ldots,[f^jX_j,f^iX_i]\ldots]]=0$. We have $[f^jX_j,[f^jX_j,\ldots,[f^jX_j,D(f^iX_i)]\ldots]]\underset{U}{\equiv}0$ and $(D(f^iX_i))_{|U}\equiv 0$. In addition, when we are unable to choose $i\neq j$, the proof is trivial. Thus, D is local.

Second, suppose that D is local. Agreeing with the result of Proposition 3.13, we achieve our proof.

Theorem 3.16. *All m-derivations of the normalizer* $\mathfrak{N}$ *of* $\mathfrak{A}_S$ *have a form like the one of Theorem 3.12. Moreover, the normalizer of* $\mathfrak{N}$ *is* $\mathfrak{N}$ *itself.*

Proof. Given $D=D_0+D_2$ an m-derivation of $\mathfrak{N}$, where D_0 is local and D_2 non-local. By Proposition 3.10, $D_{0|\mathfrak{A}_S}$ is an m-derivation of . Let be $X_1\in\mathfrak{N}$ and $X_2,\ldots,X_m\in\mathfrak{A}_S$, we write the equation relative to m-derivations corresponding to $D_0[X_1,[X_2,\ldots,[X_{m-1},X_m]\ldots]]$. By the definition of m-derivations, the previous result and Theorem 3.12, we prove that

$$\left[\left(D_0-\left(\mathrm{L}_X+D^{\alpha,\omega}+D_1\right)\right)(X_1),[X_2,\ldots,[X_{m-1},X_m]\ldots]\right]=0 \quad (3.5)$$

$\forall X_2,\ldots,X_m\in\mathfrak{A}_S$. Let's denote by D' the m-derivation defined by $D_0-(\mathrm{L}_X+D^{\alpha,\omega}+D_1)$, $D'(X_1)$ belongs to the intersection of the centralizer of $\mathfrak{A}_S^1\oplus[\mathfrak{A}_S^1,\mathfrak{A}_S^2]$ with $\mathfrak{N}$. With Proposition 3.3, $D'(X_1)$ becomes an element of $\langle S\rangle$. In addition, we know that $D_{2|\mathfrak{A}_S}=0$. By using $X_{i\neq 1}\in\mathfrak{A}_S$ in the relation of m-derivation similar to (3.5), the Proposition 3.1 and Proposition 3.3, we have $D_2(X_1)\in\langle S\rangle$. Then, Proposition 3.14 and Proposition 3.15 split D' to a sum of a derivation of $\mathfrak{A}_S$ and L_X with $X\in\langle S\rangle$, D_2 is zero. Moreover, we can affirm that the normalizer of $\mathfrak{N}$ coincides with $\mathfrak{N}$ itself.

Proposition 3.17. *Every endomorphism D of the commutative Lie algebra* $\langle S\rangle$ *is an m-derivation of* $\langle S\rangle$. *If D is local, it is a Lie derivative with respect to one element of* $\mathfrak{N}$. *In the case where D is non local, then it is determined by the existence of* $i\neq j$ *such that* $\varnothing\neq\complement\mathrm{Supp}(X_i)\subset\mathrm{Supp}(X_j)$ *with* $D(X_i)=\lambda_i^kX_k$, *where each* $\lambda_i^k\in\mathbb{R}$ *and* $\lambda_i^j\in\mathbb{R}^*$.

Proof. The first assertion is obvious. Moreover, it is clear that the normalizer of $\langle S\rangle$ is $\mathfrak{N}$, and its centralizer is $\mathfrak{A}_S$. Let D be a local endomorphism of $\langle S\rangle$. On U, we put $D\left(\frac{\partial}{\partial y^i}\right)=\lambda_i^k\frac{\partial}{\partial y^k}$. Then $D=\mathrm{L}_X$ with $X\underset{U}{=}-\lambda_i^ky^i\frac{\partial}{\partial y^k}+Y$ and $Y\in\{\mathfrak{A}\}_{S|U}$, for all U. Thus, X belongs to $\mathfrak{N}$. In addition, if D' is a non-local endomorphism of $\langle S\rangle$, let's write $D'(X_i)=\lambda_i^kX_k$. It is easy to see that D' is non-local iff our last assertion is true.

With the help of the previous proposition, we can confirm immediately

Corollary 3.18. *If all elements of S are densely supported over* M *or if S is reduced to a singleton, then all endomorphisms of* $\langle S\rangle$ *are local.*

Applications

The following is a list of some Lie algebras for which our theorems hold.

We denote by $\mathfrak{C}_c$ the Lie algebra of all compactly supported vector fields on M which is an involutive distribution over M. We know that the normalizer of $\mathfrak{C}_c$ in $\chi(M)$ is $\chi(M)$ and $O_{\mathfrak{C}_c}=M$ see [7].

We suppose that M is a differentiable manifold equipped with a nonsingular generalized foliation $\mathfrak{F}$ see [1]. We denote $\chi(\mathfrak{F})$ (resp. $\chi_c(\mathfrak{F})$ the involutive distribution of tangent vector fields to the foliation (resp. of compactly supported vector fields in $\chi(\mathfrak{F})$). The normalizer of $\chi(\mathfrak{F})$ in $\chi(M)$ is denoted $\mathfrak{N}(\mathfrak{F})$. The foliation preserving vector fields Lie algebra is named $\mathfrak{L}(\mathfrak{F})$.

Here, V is a smooth manifold and μ is surjective smooth map from M to V see [5]. It is well known that the set of μ-projected vector fields $\mathcal{N}_0$ is a Lie algebra, and μ-zero-projected vector fields set η_0 is an involutive distribution of M. The normalizer of η_0 in $\chi(M)$ is denoted $\mathcal{N}$ and we assume that $O_{\eta_0}=M$.

Now, let Γ be a connection in the Grifone sense over M cf. [9]. We can cite the curvature horizontal nullity distribution space $\mathfrak{N}_R^h$ (R is the curvature), the distribution of horizontal projected vector fields in the curvature nullity space $\mathfrak{A}_\Gamma^h$. Their respective normalizers in (TM) are designated by $\mathcal{N}_R$, $\mathcal{N}_\Gamma$ see [10] and we suppose that $O_{\mathfrak{N}_R^h}=O_{\mathfrak{A}_\Gamma^h}=TM$.

We call N^k the k-nullity space distribution of vector fields in the Finsler space considered by [11] such that the nullity index doesn't vanish everywhere. Let's note that $\mathcal{N}^k$ is its normalizer in the vector fields Lie algebra.

Thus, replacing respectively $\mathfrak{A}$ by $\chi(M)$, $\mathfrak{C}_c$, $\chi(\mathfrak{F})$, $\chi_c(\mathfrak{F})$, $\mathfrak{L}(\mathfrak{F})$, η_0, $\mathfrak{N}_R^h$, $\mathfrak{A}_\Gamma^h$, N^k and $\mathfrak{B}$ by $\chi(M)$, $\chi(M)$, $\mathfrak{N}(\mathfrak{F})$, $\mathfrak{N}(\mathfrak{F})$, $\mathfrak{N}(\mathfrak{F})$, $\mathcal{N}$, $\mathcal{N}$, $\mathcal{N}_R$, $\mathcal{N}_\Gamma$, $\mathcal{N}^k$; we state that " all m-derivations of $\mathfrak{A}$ (resp. of $\mathfrak{B}$) are inner with respect to $\mathfrak{B}$ (resp. are inner)".

In addition, let's consider the system S composed by the Liouville vector field C on TM. We work on TM without zero section, we find all m-derivations of $\mathfrak{A}_S$ by our theorem, as well as its normalizer which is locally isomorphic to $\mathfrak{A}_S \oplus gl(1,\mathbb{R})$. By density of the foliation regular points set defined by *S*, we obtain analogous results on TM. All $\mathbb{R}$-linear maps of $\langle C \rangle$ into itself are local.

The author benefits an Ingrid Daubechies initiative scholarship in collaboration with" Institute for the Conservation of Tropical Environments" (ICTE) Madagascar.

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Citation: Randriambololondrantomalala P (2015) The *m*-Derivations of Distribution Lie Algebras. J Generalized Lie Theory Appl 9: 217. doi: 10.4172/1736-4337.1000217

Randriambololondrantomalala, Princy
The m-derivations of distribution Lie algebras. (English) Zbl 06499576
J. Gen. Lie Theory Appl. 9, Article ID 1000217, 6 p. (2015).

Summary: Let M be a N-dimensional smooth differentiable manifold. Here, we are going to analyze $(m>1)$-derivations of Lie algebras relative to an involutive distribution on subrings of real smooth functions on M. First, we prove that any $(m>1)$-derivations of a distribution Ω on the ring of real functions on M as well as those of the normalizer of Ω are Lie derivatives with respect to one and only one element of this normalizer, if Ω doesn't vanish everywhere. Next, suppose that $N=n+q$ such that $n>0$, and let S be a system of q mutually commuting vector fields. The Lie algebra of vector fields $\mathfrak{A}_s$ on M which commutes with S, is a distribution over the ring $F_0(M)$ of constant real functions on the leaves generated by S. We find that m-derivations of $\mathfrak{A}_s$ is local if and only if its derivative ideal coincides with $\mathfrak{A}_s$ itself. Then, we characterize all non local m-derivation of $\mathfrak{A}_s$. We prove that all m-derivations of $\mathfrak{A}_s$ and the normalizer of $\mathfrak{A}_s$ are derivations. We will make these derivations and those of the centralizer of $\mathfrak{A}_s$ more explicit.

MSC:

17B66 Lie algebras of vector fields and related (super)algebras
17B40 Automorphisms, derivations and other operators on Lie algebras
53C12 Foliations (differential geometry)
53B15 Other connections
47B47 Commutators, derivations, elementary operators, etc.
53B40 Finsler spaces and generalizations (areal metrics)

Keywords:

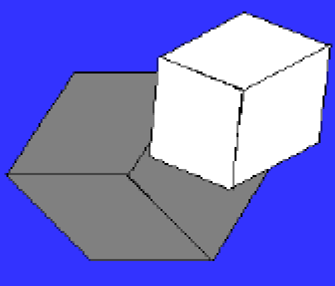
m-derivations; vector fields Lie algebras; distributions; commuting vector fields; generalized foliations; compactly supported vector fields; μ-projected vector fields; nullity space of curvature

Full Text: DOI Euclid

N° 29 – December 2012

Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227



F O R U M

Italian Journal of Pure and Applied Mathematics

ISSN 2239-0227

Web Site

http://ijpam.uniud.it/journal/home.html

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forum@forumeditrice.it

Italian Journal of Pure and Applied Mathematics – N. 29–2012

ISSN 2239-0227

SUR LES ALGEBRES DE LIE D'UN SYSTEME DE CHAMPS DE VECTEURS PERMUTABLES

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Résumé. Soient M une variété $C^\infty-$ différentiable et S un système de q $C^\infty-$ champs de vecteurs qui commutent deux à deux. Ce système définit une structure de feuilletage généralisé $\mathcal{F}$ sur M. L'algèbre de Lie A_S des champs de vecteurs de M qui commutent avec S est à la fois un module sur l'anneau des $C^\infty-$ fonctions qui sont constantes sur les feuilles de $\mathcal{F}$ et une sous-algèbre de Lie de l'algèbre de Lie des automorphismes infinitésimaux au feuilletage. On détermine toutes les dérivations de l'algèbre de Lie A_S.
Mots clés: algèbre de Lie, champ de vecteurs permutables, feuilletage généralisé, cohomologie locale de Chevalley-Eilenberg, cohomologie de de Rham.

Abstract. Let be M a $C^\infty-$ differentiable manifold and S a system of q $C^\infty-$ vector fields which commute mutually. This system defines a generalized foliation $\mathcal{F}$ on M. The Lie algebra A_S of vector fields in M which commute with S is both a module over the ring of $C^\infty-$ functions that are constant on the leaves of $\mathcal{F}$ and a sub-Lie algebra of the foliation preserving vector fields. We determine all derivations of the Lie algebra A_S.

Keywords: Lie Algebra, commuting vector fields, generalized foliation, local cohomology of Chevalley-Eilenberg, cohomology of de Rham.

AMS Subject Classification: Primary 17B66; 17B56; Secondary 53C12; 47B47.

1. Introduction

Soient M une variété différentiable paracompacte de classe C^∞ et $\chi(\mathrm{M})$ l'algèbre de Lie des champs de vecteurs de M. Dans son article [8], Takens a montré que toute dérivation de l'algèbre de Lie $\chi(\mathrm{M})$ est une dérivée de Lie par rapport à un champ de vecteurs de M. Dans le cas où l'algèbre de Lie est une sous-algèbre de Lie attachée à un feuilletage régulier sur M, Lichnérowicz cf. [3] a prouvé aussi des résultats analogues. Nous avons étendu ces résultats dans le cas d'une distribution involutive non régulière cf. [5], où l'anneau de base contient toutes les fonctions de classe C^∞ de la variété. Dans [6], nous avons abordé le même problème sur les algèbres de Lie des champs de vecteurs polynomiaux $\mathfrak{P}$ sur $\mathbb{R}^n$ qui contiennent tous les champs constants et le champ d'Euler. Nous avons prouvé que toute dérivation de $\mathfrak{P}$ est une dérivée de Lie par rapport à un champ de vecteurs polynomiaux de $\mathbb{R}^n$. Dans ce papier, nous étudions une sous-algèbre de Lie de $\chi\,(\mathrm{M})$ dont l'anneau des fonctions de classe C^∞ du module sous-jacent est tronqué. Plus précisement, M est une variété différentiable de dimension $m+q$ et S un système de $q \geq 1$ champs de vecteurs qui commutent deux à deux, et de rang p avec $0 \leq p\,(x) \leq q$, pour tout $x \in \mathrm{M}$. Il existe une structure de feuilletage généralisé $\mathfrak{F}$ définie par le système S cf. [1]. On note $L_{\mathfrak{F}}$ l'algèbre de Lie des champs des automorphismes infinitésimaux de $\mathfrak{F}$ et, A_S l'algèbre de Lie des champs de vecteurs de M qui commutent avec S. Toutes les feuilles sont supposées régulières. L'algèbre de Lie A_S se décompose en une somme semi-directe d'algèbres de Lie A_S^1 et A_S^2, où A_S^1 (resp. A_S^2) est un module (resp. l'algèbre de Lie engendrée par S) sur l'anneau des fonctions constantes aux feuilles. Ainsi A_S est une sous-algèbre de Lie de $L_{\mathfrak{F}}$. De plus, l'algèbre de Lie A_S^2 est un idéal caractéristique de A_S. Par ailleurs, on donne une condition nécéssaire et suffisante pour que toute dérivation de A_S soit locale; de même pour que l'idéal dérivé de A_S coïncide à A_S. Ainsi, les caractéristiques d'une dérivation non locale de A_S sont obtenues. En étudiant la dérivation locale de A_S dans l'idéal caractéristique A_S^2, on peut déterminer toutes les dérivations locales non intérieures de A_S. Par suite, en utilisant l'algèbre quotient de A_S par A_S^2 et un résultat de [5], on peut décomposer toute dérivation locale de A_S en une somme de dérivation intérieure de A_S et de dérivation locale non intérieure trouvée auparavant. Dans le cas où le rang p de S est constant supérieur ou égal à 1, le premier espace de cohomologie locale de Chevalley-Eilenberg de A_S est isomorphe à $(\mathrm{H}^1_R\,(\mathrm{B}) \times \mathbb{R})^p \times \mathbb{R}^{p^2}$, où $\mathrm{H}^1_R\,(\mathrm{B})$ désigne le premier espace de cohomologie de de Rham sur les formes basiques au feuilletage de M. Si le système S est réduit à un champ de Liouville, on retrouve par une méthode différente un résultat de Lecomte dans [4].

2. Préliminaires

Soit M une variété réelle $C^\infty-$ différentiable paracompacte de dimension $m+q$ où $m, q \geq 1$. Tous les objets étudiés sont supposés de classe C^∞. On désigne par $F\,(\mathrm{M})$ l'anneau des fonctions C^∞ sur M, $\chi\,(\mathrm{M})$ l'algèbre de Lie des champs de vecteurs sur M, S un système $\{X_1, \ldots, X_q\}$ de rang p de champs de vecteurs, avec $0 \leq p\,(x) \leq q$ pour tout $x \in \mathrm{M}$. Les éléments de S vérifient $[X_i, X_j] = 0$ pour tous $i, j \in \{1, \ldots, q\}$. On considère l'algèbre de Lie A_S des champs de vecteurs X de M tels que $[X, X_i] = 0$ pour tout $i \in \{1, \ldots, q\}$.

On peut déduire du système S un champ de plans P, qui à tout $x \in \mathrm{M}$ correspond le sous-espace vectoriel engendré par $X_1(x), \ldots, X_q(x)$ de $T_x\,(\mathrm{M})$. P est un champ de plans de classe C^∞ de système générateur S. Tout champ de vecteurs $X = g^j X_j$ de P avec $g^j \in F(\mathrm{M})$, vérifie pour tout i

$$[X, X_i] = -\left(X_i(g^j)\right) X_j$$

c'est-à-dire, P est invariant par tout champ de vecteurs de P. D'après le théorème de Sussmann cf. [7], il existe un feuilletage généralisé $\mathfrak{F}$ sur M dont la feuille en un point x de M est la variété intégrale maximale $I(x)$ telle que pour tout $y \in I(x)$,

$T_y\left(I(x)\right) = P_y$ cf. [1]. On note $F_0\,(\mathrm{M})$ l'anneau des fonctions sur M constantes aux feuilles. La sous-algèbre de Lie A_S^2 des champs de vecteurs de M engendrée par S sur $F_0\,(\mathrm{M})$ est commutative. De plus A_S^2 est une sous-algèbre de Lie de l'algèbre de Lie L des champs de vecteurs tangents aux feuilles. Par ailleurs, A_S^1 désigne l'ensemble des champs de vecteurs de M tel que A_S^1 et L sont deux sous-modules supplémentaires dans l'algèbre de Lie $L_{\mathfrak{F}}$ des automorphismes infinitésimaux au feuilletage.

On suppose que toutes les feuilles soient régulières, sauf mention expresse. Le théorème de Dazord cf. [1] p.415 assure l'existence d'une carte adaptée (U, x^a, y^i) (resp. (U, x^a)), avec $1 \leq a \leq m+q-p$, $1 \leq i \leq p$ si $p \geq 1$ (resp. $1 \leq a \leq m+q$ si $p = 0$) au voisinage de chaque point x de M où la dimension de $I(x)$ est constante $p(x) = p$. Il existe une permutation ζ de $\{1, \ldots, q\}$ tels que pour $p \geq 1$ (resp. $p = 0$) $\left(X_{\zeta_i} \underset{U}{=} \frac{\partial}{\partial y^i}\right)_{1 \leq i \leq p}$ et $\left(X_{\zeta_l} \underset{U}{=} 0\right)_{p<l\leq q}$ (resp. $\left(X_l \underset{U}{=} 0\right)_{1\leq l\leq q}$). On utilisera de tels ouverts pour les domaines de cartes adaptées au feuilletage. On conviendra dans la suite sauf mention expresse que les indices a, b, c vont de 1 à $m+q-p$ et i, j, l de 1 à p si $p \geq 1$. De même, les indices fixes a_0, a_1, b_0 appartiennent à $\{1, \ldots, m+q-p\}$ et i_0, j_0 à $\{1, \ldots, p\}$ si $p \geq 1$.

L'anneau $F_0\,(U) = \left\{f_{|U} \text{ tel que } f \in F_0\,(\mathrm{M})\right\}$ est l'ensemble des fonctions sur U ne dépendant pas des coordonnées y^i. L'algèbre de Lie A_S sur toute carte adaptée U, coïncide au $F_0\,(U)$-module des champs sur U engendré par $\frac{\partial}{\partial x^1}, \ldots,$ $\frac{\partial}{\partial x^{m+q-p}}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^p}$ où $p \geq 1$. Le module $A_S\,(U)$ se décompose en produit semi-direct

$$A_S\,(U) = A_S^1\,(U) \oplus A_S^2\,(U)$$

où $A_S^1\,(U)$ est la sous-algèbre de $A_S\,(U)$ engendrée par $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m+q-p}}$ sur $F_0\,(U)$ et, où $A_S^2\,(U)$ est l'idéal commutatif de $A_S\,(U)$ engendré par $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^p}$ sur $F_0\,(U)$.

Dans le cas où $p = 0$, $F_0\,(U) = F\,(U)$ et

$$A_S\,(U) = A_S^1\,(U) \oplus A_S^2\,(U)$$

avec $A_S^1\,(U) = \mathcal{X}(U)$ et $A_S^2\,(U) = \{0\}$.

On s'interesse à l'étude des $\mathbb{R}-$dérivations de l'algèbre de Lie A_S. Le cas trivial où le rang est identiquement nul sur M, est déjà étudié par [8]. Donc, on suppose que $S \neq \{0\}$.

Remarque 2.1 Si la variété M est connexe, le feuilletage défini est régulier d'après une assertion de [1] p.416.

3. Etude des dérivations de A_S

Dans toute la suite $x \in$ M est un point quelconque, U est une carte adaptée contenant x telle que la dimension de $I(x)$ est une constante égale à p sur U. On utilisera la convention d'Einstein sur la sommation d'indices, sauf mention expresse.

Définition 3.1 Le centralisateur (resp. Le centre) de A_S est l'ensemble des X dans $\mathfrak{X}(\mathrm{M})$ (resp. dans A_S) tels que $[X, A_S] = \{0\}$.

Proposition 3.2 *Le centralisateur $\mathfrak{C}$ de A_S est le $\mathbb{R}-$espace vectoriel engendré par S.*

Démonstration. Il est immédiat que le $\mathbb{R}-$espace vectoriel engendré par S est inclus dans $\mathfrak{C}$.

Réciproquement, soit X appartenant à $\mathfrak{C}$. Sur U, si $p = 0$, alors la preuve est donnée par un résultat de [8]. Si $p \geq 1$, soit $X_{|U} = X^a \frac{\partial}{\partial x^a} + X'^i \frac{\partial}{\partial y^i} \in \mathfrak{C}_U$, on a

$$\left[X^a \frac{\partial}{\partial x^a} + X'^i \frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^b}\right] = 0, \; \left[X^a \frac{\partial}{\partial x^a} + X'^i \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0$$

pour tous b, j. Ainsi, chaque X^a et X'^i sont des constantes réelles en supposant que U est connexe. Par ailleurs,

$$\left[X^a \frac{\partial}{\partial x^a} + X'^i \frac{\partial}{\partial y^i}, x^c \frac{\partial}{\partial x^c}\right] = 0$$

alors on en déduit que chaque $X^a = 0$ pour tous $X \in \mathfrak{C}$ et U adaptée à $\mathfrak{F}$. Donc $\mathfrak{C}$ est contenu dans le $\mathbb{R}-$espace vectoriel engendré par S. D'où le résultat. ■

Remarque 3.3 Le système S n'est pas en général une base du centre de A_S. Par exemple, sur le tore T^2 avec $S = \{X\}$, où X est un champ de vecteurs invariant ayant une trajectoire dense.

Définition 3.4 Une $\mathbb{R}$-dérivation D d'une sous-algèbre de Lie $\mathfrak{A}$ des champs de vecteurs sur M, est une application $\mathbb{R}-$linéaire de $\mathfrak{A}$ dans $\mathfrak{A}$ telle que

$$\forall X, Y \in \mathfrak{A}, \; D[X, Y] = [D(X), Y] + [X, D(Y)]. \tag{3.1}$$

L'application D est dite dérivation intérieure de $\mathfrak{A}$ si $D = [X, .] = \mathrm{L}_X$, avec L_X la dérivée de Lie par rapport à $X \in \mathfrak{A}$.

Dans cette section, une $\mathbb{R}$-dérivation d'une algèbre de Lie $\mathfrak{A}$ est tout simplement appelée dérivation de $\mathfrak{A}$.

Proposition 3.5 *Soient D une dérivation de A_S et U un domaine d'une carte adaptée de* M *tels qu'il existe $X \in A_S$ avec $X_{|U} \equiv 0$, alors $(D(X))_{|A_S^1} \underset{U}{\equiv} 0$*

Démonstration. On considère une dérivation D de A_S. On suppose que $X \in A_S$ et $X_{|U} \equiv 0$. On peut écrire $(D(X))_{|A_S^1} \underset{U}{=} D_X^a \frac{\partial}{\partial x^a}$. Si $D(X)$ est non identiquement nul sur $A_S^1(U)$, il existe un point $z \in U$ tel que l'une au moins des composantes correspondantes de $D(X)$ soit non nulle en z. On suppose qu'il existe un entier a_0 tel que $D_X^{a_0}(z) \neq 0$, donc on peut trouver un ouvert V_z contenant z tel que $D_X^{a_0}(y) \neq 0$ pout tout $y \in V_z$. On prend $f \in F_0(\mathrm{M})$ où $f_{|V_z} = x^{a_0}$ avec

$\operatorname{supp}(f) \subset U$ et, $Y \in A_S$ tel que $Y_{|V_z} = \frac{\partial}{\partial y^{i_0}}$. De cette façon, $[X, fY]_{|U} \equiv 0$ et $[X, fY]_{|\complement U \subset \complement \operatorname{supp}(f)} \equiv 0$ et $[X, fY] \equiv 0$. Ainsi, la relation suivante

$$D([X, fY]) = [D(X), fY] + [X, D(fY)] \tag{3.2}$$

aboutit à une contradiction. D'où le résultat. ∎

Définition 3.6 Soient $\mathfrak{A}$ et $\mathfrak{B}$ deux sous-modules d'une même algèbre de Lie. Le sous-module engendré par tous les crochets de $X \in \mathfrak{A}$ et $Y \in \mathfrak{B}$ est noté par $[\mathfrak{A}, \mathfrak{B}]$. Si $\mathfrak{A} = \mathfrak{B}$ et que $\mathfrak{A}$ est une algèbre de Lie, alors on l'appelle idéal dérivé de $\mathfrak{A}$.

Proposition 3.7 *L'idéal dérivé de A_S^1 est égal à A_S^1, l'idéal dérivé de A_S^2 est nul. Ainsi, l'idéal dérivé de A_S coïncide à la somme directe de module A_S^1 et de l'algèbre de Lie A engendrée par les $[X, Y]$ où $X \in A_S^1$ et $Y \in A_S^2$.*

Démonstration. On peut adapter la preuve de la Proposition 2.9 p.141 de [5] pour avoir $[A_S^1, A_S^1] = A_S^1$. Ainsi, l'idéal dérivé de A_S^1 coïncide à A_S^1. Par ailleurs, il est clair que $[A_S^2, A_S^2]$ est réduit à $\{0\}$. Comme $[A_S, A_S] = [A_S^1 \oplus A_S^2, A_S^1 \oplus A_S^2]$, alors cette dérnière devient $A_S^1 \oplus [A_S^1, A_S^2]$ avec $\oplus$ désigne une somme directe de modules, d'où le résultat. ∎

Dans la suite, on note $A = [A_S^1, A_S^2]$.

Proposition 3.8 *Toute dérivation non locale de A_S est à la fois à valeur dans le centre de A_S, nulle sur A_S^1 et sur $A \subset A_S^2$.*

Démonstration. Soit D une dérivation non locale de A_S, alors on peut trouver $X \in A_S$ et U un ouvert de M tel que $X_{|U} \equiv 0$ avec $D(X)$ n'est pas nul sur U. Donc il existe un ouvert $W \subset U$ contenant $x \in$ M tel que $D(X)(y) \neq 0$ pour tout $y \in W$. D'après la Proposition 3.5, on écrit $D(X) \underset{U}{=} D_X^i X_i$. Il s'en suit qu'il existe i_0 tel que $D_X^{i_0}(y) \neq 0$ pour tout y dans un ouvert $W' \subset W$ contenant x. Supposons que $D(X)$ n'appartient pas à $\mathfrak{C}$, alors on peut supposer que $\left(D_X^{i_0}\right)_{|W'}$ est non constante. On prend $f \in F_0(\mathrm{M})$ tel que $\operatorname{supp}(f) \subset U$ avec $f(x) \neq 0$. Aussi, peut-on trouver $Y \in A_S^1$ tel que $Y_{|U} = \frac{\partial}{\partial x^{a_0}}$ de façon que $\frac{\partial D_X^{i_0}}{\partial x^{a_0}}(x) \neq 0$. Dans ce cas, une relation analogue à celle de (3.2) donne une contradiction au point x. Par conséquent, $D(X) \in \mathfrak{C}$. On déduit du résultat qui précède et de la propriété (3.1) d'une dérivation que $D[A_S, A_S] = \{0\}$. Par la Proposition 3.7, on en tire que $D(A_S^1) = \{0\}$ et $D(A) = \{0\}$. ∎

Proposition 3.9 *L'algèbre de Lie A_S^2 est stable par toute dérivation locale de A_S.*

Démonstration. Soit D une dérivation locale de A_S, D_U est une dérivation de $A_S(U)$ en faisant le même raisonnement que celui de [8] p.157. Sur U, si $p = 0$ alors la preuve est évidente. Sur cet ouvert, si $p \geq 1$ alors l'algèbre de Lie A_S s'écrit $A_S(U) = A_S^1(U) \oplus A_S^2(U)$. Or, chaque $\frac{\partial}{\partial y^i}$ est un élément du centre de $A_S(U)$ et que le centre d'une algèbre de Lie est un idéal caractéristique, alors $D_U\left(\frac{\partial}{\partial y^i}\right)$ appartient au centre qui est contenu dans $A_S^2(U)$ par la Proposition 3.2. Pour chaque a, b, de la relation $\left[\frac{\partial}{\partial x^a}, x^b\frac{\partial}{\partial y^i}\right] = \delta_a^b\frac{\partial}{\partial y^i}$, on obtient $\delta_a^b D_U\left(\frac{\partial}{\partial y^i}\right) = \left[D_U\left(\frac{\partial}{\partial x^a}\right), x^b\frac{\partial}{\partial y^i}\right] + \left[\frac{\partial}{\partial x^a}, D_U\left(x^b\frac{\partial}{\partial y^i}\right)\right]$. Or $A_S^2(U)$ est un idéal de $A_S(U)$, alors $\left[\frac{\partial}{\partial x^a}, D_U\left(x^b\frac{\partial}{\partial y^i}\right)\right] \in A_S^2(U)$. Ainsi, en posant $D_U\left(x^b\frac{\partial}{\partial y^i}\right) = D_{i,b}^c\frac{\partial}{\partial x^c} + D_{i,b}^{j+m+q-p}\frac{\partial}{\partial y^j}$, chaque $D_{i,b}^c$ est constant. D'autre part, en appliquant D_U à l'égalité $x^b\frac{\partial}{\partial y^i} = \left[x^c\frac{\partial}{\partial x^c}, x^b\frac{\partial}{\partial y^i}\right]$, il s'en suit que $D_U\left(x^b\frac{\partial}{\partial y^i}\right) + \left[D_U\left(x^b\frac{\partial}{\partial y^i}\right), x^c\frac{\partial}{\partial x^c}\right]$ appartient à $A_S^2(U)$. Comme chaque $D_{i,b}^c$ est constant, on obtient $D_{i,b}^c = 0$. Autrement dit, $D_U\left(x^b\frac{\partial}{\partial y^i}\right)$ est un élément de $A_S^2(U)$. En dérivant par D_U la relation $\left[f\frac{\partial}{\partial x^a}, x^a\frac{\partial}{\partial y^i}\right] = f\frac{\partial}{\partial y^i}$ pour tout $f \in F_0(U)$, on trouve que $D_U\left(f\frac{\partial}{\partial y^i}\right)$ est encore dans $A_S^2(U)$. Sachant que tout élément de $A_S^2(U)$ est engendré par les $\frac{\partial}{\partial y^i}$ sur les fonctions de $F_0(U)$, toute dérivation D_U de $A_S(U)$ préserve l'idéal $A_S^2(U)$. D'où le résultat. ∎

Proposition 3.10 *L'algèbre de Lie A_S^2 est un idéal caractéristique commutatif de A_S.*

Démonstration. Soit D une dérivation de A_S, D est la somme d'une dérivation locale D_0 et d'une dérivation non locale D_1 de A_S. D'après la Proposition 3.9, on a $D_0(A_S^2) \subset A_S^2$; et de la Proposition 3.8, $D_1(A_S) \subset A_S^2$. En utilisant la $\mathbb{R}$-linéarité de D et $[A_S^2, A_S^2] = \{0\}$, on a le résultat. ∎

Théorème 3.11 *On suppose que pour tout $x \in \mathrm{M}$, $0 < p(x) < q$. Les assertions suivantes sont équivalentes:*

1. *Toute dérivation de A_S est locale.*
2. *Il existe $X \in A_S^1$ et $h \in F_0(\mathrm{M})$ tels que $X(h)$ est partout non nul sur* M.
3. *L'idéal dérivé de A_S est A_S.*

Démonstration. (1.) ⇔ (2.): Soit D une dérivation de A_S, D est la somme d'une dérivation locale et d'une dérivation non locale de A_S. On note alors D_1 cette dérivation non locale. Etant donné un $X \in A_S^2 - \{0\}$, on calcule $D_1(X)$. Par le fait que D_1 soit $\mathbb{R}-$linéaire, on peut supposer seulement qu'il existe $f \in F_0(\mathrm{M})$ et i_0 tels que $X = fX_{i_0}$. On écrit $D_1(X) = D_f^i X_i$; avec les $D_f^i \in \mathbb{R}$ d'après la Proposition 3.8 et la Proposition 3.2. Soient $Y \in A_S^1$, $g \in F_0(\mathrm{M})$ tels que $Y(g)$ est partout non nul, et $h \in F_0(\mathrm{M})$. Comme $D_1[hY, gX_{i_0}] = 0$ d'après la Proposition 3.8, alors chaque $D_{hY(g)}^i = 0$. Or, on peut trouver h tel que $hY(g) = f$ en posant $h = \frac{f}{Y(g)} \in F_0(\mathrm{M})$. Ainsi, $D_1(X) = 0$ pour tout $X \in A_S^2$ et par conséquent, $D_1 = 0$ car $D_{1|A_S^1} = 0$ d'après la Proposition 3.8. C'est-à-dire que toute dérivation D de A_S est locale. Réciproquement, soit D l'application $\mathbb{R}$-linéaire définie par

$$D(X) = \begin{cases} 0 & \text{si } X \in A_S - \mathfrak{C}, \\ \sum\limits_{1\leq j\leq q} \alpha^j \sum\limits_{1\leq k\leq q} X_k & \text{si } X = \alpha^i X_i \text{ où } \alpha^j \in \mathbb{R} \text{ pour tout } j = 1, \ldots, q. \end{cases}$$

En supposant que quel que soit $x \in$ M, $p(x) < q$; il existe i_0 dans $\{1, \ldots, q\}$ et un ouvert U de M, tels que $X_{i_0|U} \equiv 0$. Ainsi, on a $D(X_{i_0}) = \sum_{1 \leq k \leq q} X_k$ tel que, $D\left(X_{i_0}\right)_{|U}$ est non nul, car pour tout $x \in$ U, $p(x) > 0$. Si pour tous $X \in A_S^1$ et $h \in F_0\,(\mathrm{M})$, il existe $x \in$ M tels que $X\,(h)\,(x) = 0$, alors $fX\,(h) = 1$ est impossible, quel que soit $f \in F_0\,(\mathrm{M})$. Alors, $[A_S^1, A_S^2]$ ne contient pas d'éléments de $\mathfrak{C} - \{0\}$, et on a $D\,(A) = \{0\}$. Ainsi, D est une dérivation non locale de A_S.

(2.) $\Leftrightarrow$ (3.): Si l'idéal dérivé de A_S est A_S, alors toute dérivation non locale de A_S est nulle, d'après la Proposition 3.8. Ainsi, toute dérivation de A_S est locale. D'après (1.) $\Rightarrow$ (2.), on a le résultat. Réciproquement, la deuxième partie de la preuve de (1.) $\Leftrightarrow$ (2.) permet de conclure. ■

Remarque 3.12 On peut omettre l'hypothèse "pour tout $x \in$ M, $0 < p(x) < q$" en prouvant (2.) $\Rightarrow$ (1.), et (2.) $\Rightarrow$ (3.) du Théorème 3.11.

Remarque 3.13 Si quels que soient $f,\ h \in F_0\,(\mathrm{M})$ et $X \in A_S^1$, $fX\,(h) \neq 1$, la réciproque de la Proposition 3.8 est fausse car la dérivation D de A_S définie par

$$D(X) = \begin{cases} 0 & \text{si } X \in A_S - \mathfrak{C}, \\ \alpha^k X_k & \text{si } X = \alpha^k X_k \text{ où } \alpha^k \in \mathbb{R} \text{ pour tout } k = 1, \ldots, q. \end{cases}$$

est une dérivation locale. Pourtant, D vérifie toutes les conditions nécéssaires de cette proposition.

Dans les trois propositions suivantes, on suppose que $p \geq 1$ sur U.

Proposition 3.14 *Soit D une dérivation locale de A_S dans A_S^2. Si $D_U = \beta^j \otimes \frac{\partial}{\partial y^j}$ où chaque β^i est une forme linéaire de $A_S\,(U)$ dans $F_0\,(U)$, alors β^i est fermée. De plus, si chaque β^i s'annule sur $A_S^2\,(U)$, alors pour tous $X, Y \in A_S(U)$, on a pour tout i*

$$\beta^i\,[X, Y]\,\frac{\partial}{\partial y^i} = \left[\beta^i\,(X)\,\frac{\partial}{\partial y^i}, Y\right] + \left[X, \beta^i\,(Y)\,\frac{\partial}{\partial y^i}\right]. \tag{3.3}$$

Démonstration. On prend $i \in \{1, \ldots, p\}$, β^i est de la forme $\beta^i = \beta_a^i dx^a + \beta'^i_j dy^j$ où chaque $\beta_a^i, \beta'^i_j \in F_0\,(U)$. Soient $X, Y \in A_S\,(U)$, par la propriété d'une dérivation, on obtient

$$D_U\,[X, Y] = [D_U\,(X)\,, Y] + [X, D_U\,(Y)] \tag{3.4}$$

En posant $X = X^a \frac{\partial}{\partial x^a} + X'^j \frac{\partial}{\partial y^j}$ et $Y = Y^a \frac{\partial}{\partial x^a} + Y'^j \frac{\partial}{\partial y^j}$, alors on doit avoir

$$D_U\,[X, Y] = \beta_b^i X^a \frac{\partial Y^b}{\partial x^a} \frac{\partial}{\partial y^i} - \beta_b^i Y^a \frac{\partial X^b}{\partial x^a} \frac{\partial}{\partial y^i} + \beta'^i_j X^a \frac{\partial Y'^j}{\partial x^a} \frac{\partial}{\partial y^i} - \beta'^i_j Y^a \frac{\partial X'^j}{\partial x^a} \frac{\partial}{\partial y^i}$$

Le second membre de (3.4) devient

$$\begin{aligned}(3.5)\qquad & Y^a\frac{\partial\beta^i_b}{\partial x^a}X^b\frac{\partial}{\partial y^i}-Y^a\frac{\partial X^b}{\partial x^a}\beta^i_b\frac{\partial}{\partial y^i}-Y^a\frac{\partial{\beta'}^i_j}{\partial x^a}X'^j\frac{\partial}{\partial y^i}-Y^a\frac{\partial X'^j}{\partial x^a}{\beta'}^i_j\frac{\partial}{\partial y^i}\\ &+X^a\frac{\partial{\beta'}^i_j}{\partial x^a}Y'^b\frac{\partial}{\partial y^i}+X^a\frac{\partial Y'^j}{\partial x^a}{\beta'}^i_j\frac{\partial}{\partial y^i}\end{aligned}$$

Par identification membre à membre, on a

$$(3.6)\qquad -Y^a\frac{\partial\beta^i_b}{\partial x^a}X^b\frac{\partial}{\partial y^i}-Y^a\frac{\partial{\beta'}^i_j}{\partial x^a}X'^j\frac{\partial}{\partial y^i}+X^a\frac{\partial\beta^i_b}{\partial x^a}Y^b\frac{\partial}{\partial y^i}+X^a\frac{\partial{\beta'}^i_j}{\partial x^a}Y'^j\frac{\partial}{\partial y^i}=0$$

Par ailleurs, β^i est fermée si et seulement si

$$\mathrm{d}\beta^i=\left(\frac{\partial\beta^i_a}{\partial x^b}-\frac{\partial\beta^i_b}{\partial x^a}\right)dx^b\underset{a<b}{\wedge}dx^a+\left(\frac{\partial{\beta'}^i_j}{\partial x^a}\right)dx^a\wedge dy^j=0$$

où d désigne la différentielle extérieure. C'est-à-dire $\frac{\partial\beta^i_a}{\partial x^b}-\frac{\partial\beta^i_b}{\partial x^a}=0$ et $\frac{\partial{\beta'}^i_j}{\partial x^a}=0$ quels que soient j,a,b.

On prend $a_0,\ b_0$ avec $Y^{a_0}=X^{b_0}=1$, $X'^j=Y'^j=0$ pour tout j, et les autres nuls dans la relation (3.6). Ainsi, $\frac{\partial\beta^i_{a_0}}{\partial x^{b_0}}-\frac{\partial\beta^i_{b_0}}{\partial x^{a_0}}=0$, pour toute valeur arbitraire de a_0,b_0.

Soient $a_1,\ j_0$ avec $Y^{a_1}=X'^{j_0}=1$, tous les autres sont nuls et, $Y'^j=X^a=0$ pour tous j,a dans (3.6). On a alors $\frac{\partial{\beta'}^i_{j_0}}{\partial x^{a_1}}=0$, pour chaque valeur arbitraire de a_1,j_0. D'où la forme β^i est fermée.

Si β^i s'annule sur $A^2_S(U)$, alors $\beta^i=\beta^i_a dx^a$, pour tout a. On a

$$\begin{aligned}&\beta^i\left[X,Y\right]\frac{\partial}{\partial y^i}\\ (3.7)\qquad &=\beta^i\left(X^a\frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial x^b}+X^a\frac{\partial Y''^l}{\partial x^a}\frac{\partial}{\partial y^l}-Y^a\frac{\partial X^b}{\partial x^a}\frac{\partial}{\partial x^b}-Y^a\frac{\partial X''^l}{\partial x^a}\frac{\partial}{\partial y^l}\right)\frac{\partial}{\partial y^i}\ .\\ &=\beta^i_aX^b\frac{\partial Y^a}{\partial x^b}\frac{\partial}{\partial y^i}-\beta^i_aY^b\frac{\partial X^a}{\partial x^b}\frac{\partial}{\partial y^i}\end{aligned}$$

De plus,

$$\begin{aligned}(3.8)\qquad &\left[\beta^i\left(X\right)\frac{\partial}{\partial y^i},Y\right]+\left[X,\beta^i\left(Y\right)\frac{\partial}{\partial y^i}\right]=-Y^a\frac{\partial\beta^i_a}{\partial x^a}X^b\frac{\partial}{\partial y^i}-Y^a\beta^i_b\frac{\partial X^b}{\partial x^a}\frac{\partial}{\partial y^i}\\ &\qquad+X^a\frac{\partial\beta^i_b}{\partial x^a}Y^b\frac{\partial}{\partial y^i}+X^a\beta^i_b\frac{\partial Y^b}{\partial x^a}\frac{\partial}{\partial y^i}.\end{aligned}$$

Comme la forme β^i est fermée, alors $\frac{\partial\beta^i_a}{\partial x^b}-\frac{\partial\beta^i_b}{\partial x^a}$ est nul quels que soient a,b. Par conséquent, $X^a\frac{\partial\beta^i_b}{\partial x^a}Y^b\frac{\partial}{\partial y^i}=Y^a\frac{\partial\beta^i_b}{\partial x^a}X^b\frac{\partial}{\partial y^i}$. Ainsi, en identifiant (3.7) et (3.8); on obtient le résultat (3.3). ∎

Proposition 3.15 *Soit D une dérivation locale de A_S vers A_S^2. Il existe des $1-$formes différentielles fermées α^i et ω^i dans U, avec $i = 1, \ldots, p$ telles que:*

1. $D_U = (\alpha^j + \omega^j) \otimes \frac{\partial}{\partial y^j}$, *où chaque $ker(\alpha^j)$ contient $A_S^2(U)$ et chaque $ker(\omega^j)$ contient $A_S^1(U)$.*
2. *chaque $\alpha^i [X, Y] = X.\alpha^i (Y) - Y.\alpha^i (X)$, pour tous champs $X, Y \in A_S(U)$.*

On notera $D_U^{\alpha,\omega}$ la dérivation $(\alpha^j + \omega^j) \otimes \frac{\partial}{\partial y^j}$ de $A_S (U)$ vers $A_S^2 (U)$.

Démonstration. Soit $D : A_S \longrightarrow A_S^2$ une dérivation locale de l'algèbre de Lie A_S, donc la restriction $D_U : A_S(U) \longrightarrow A_S^2(U)$ l'est aussi. La dérivation D_U étant une application $\mathbb{R}-$linéaire de $A_S(U)$ vers $A_S^2(U)$. D_U doit s'écrire sous la forme

$$D_U = \beta^j \otimes \frac{\partial}{\partial y^j}$$

où les β^i sont des formes linéaires de $A_S(U)$ sur $F_0(U)$.

L'algèbre de Lie A_S^2 étant un idéal caractéristique commutatif de A_S d'après la Proposition 3.9, la restriction de D sur A_S^2 est donc une dérivation de A_S^2. Alors $D_{U|A_S^2} = \omega^j \otimes \frac{\partial}{\partial y^j}$, où ω^i sont des formes linéaires de $A_S(U)$ dans $F_0(U)$. En vertu de la Proposition 3.14, les formes β^i et ω^i sont fermées. Les formes ω^i peuvent se décomposer en $\omega^i = \omega^i_{|A_S^1(U)} + \omega^i_{|A_S^2(U)}$. En posant $\alpha^i = \beta^i - \omega^i$, les formes α^i s'annulent sur $A_S^2(U)$ pour tout i. On peut choisir α^i pour que chaque $\omega^i_{|A_S^1(U)}$ soit nulle. D'où l'assertion 3.15..
Comme $\alpha^i = \beta^i - \omega^i$, alors chaque forme α^i est fermée. Par le fait que les α^i soient fermées, pour tous $X, Y \in A_S(U)$, on a l'égalité suivante pour tout j

$$\alpha^j [X, Y] \frac{\partial}{\partial y^j} = \left[\alpha^j (X) \frac{\partial}{\partial y^j}, Y\right] + \left[X, \alpha^j (Y) \frac{\partial}{\partial y^j}\right]$$

d'après la Proposition 3.14. D'où l'assertion 3.15.

Réciproquement, il est immédiat de constater qu'une application D_U de $A_S(U)$ dans $A_S^2(U)$ vérifiant les assertions 3.15. et 3.15. est une dérivation de $A_S(U)$. ∎

Proposition 3.16 *La dérivation $D_U^{\alpha,\omega}$ de $A_S (U)$ vers $A_S^2 (U)$ de la Proposition 3.15, avec $\alpha = (\alpha^1, \ldots, \alpha^p)$ et $\omega = (\omega^1, \ldots, \omega^p)$ est intérieure si et seulement si, pour tout i, $\omega^i \equiv 0$ et α^i sont des formes exactes. Dans ce cas, on a $D_U^{\alpha,\omega} = -\mathrm{L}_{f^i \frac{\partial}{\partial y^i}}$, où chaque $\alpha^i = df^i$ avec f^i sont des fonctions de $F_0(U)$.*

Démonstration. On suppose que $D_U^{\alpha,\omega} = \mathrm{L}_Y$ avec $Y = Y'^i \frac{\partial}{\partial y^i} \in A_S^2(U)$. Pour simplifier les notations, on prend $\alpha^i = \alpha^i_j dx^j$ et $\omega^i = \omega^i_j dy^j$, pour chaque i.

Soit $X = (X^1, \ldots, X^{m+q-p}, X'^1, \ldots, X'^p)$ un élément de $A_S(U)$, or $D_U = (\alpha^i + \omega^i) \otimes \frac{\partial}{\partial y^i}$ donc

$$\begin{aligned} D_U (X) &= \left(\left(\alpha^i + \omega^i\right) \otimes \frac{\partial}{\partial y^i}\right)(X) = \left(X^j \alpha^i_j + X'^j \omega^i_j\right) \frac{\partial}{\partial y^i} \\ &= \left[Y'^i \frac{\partial}{\partial y^i}, X^j \frac{\partial}{\partial x^j} + X'^j \frac{\partial}{\partial y^j}\right] = -X^j \frac{\partial Y'^i}{\partial x^j} \frac{\partial}{\partial y^i}. \end{aligned}$$

On a pour tout i

$$X^j\alpha_j^i + X'^j\omega_j^i = -X^j\frac{\partial Y'^i}{\partial x^j} \tag{3.9}$$

On pose dans (3.9) $X^j = 0$ quel que soit j et, $X'^j = 1$ pour un j fixé, avec $X'^l = 0$ pour $l \neq j$; on obtient $\omega_j^i = 0$ quel que soit i.

Maintenant, on pose dans (3.9) $X^j = 1$ pour j fixé, avec $X^l = 0$ pour $l \neq j$, on a $\alpha_j^i = -\frac{\partial Y'^i}{\partial x^j}$ quel que soit i. Ainsi chaque $\alpha^i = -\frac{\partial Y'^i}{\partial x^j}dx^j = df^i$ et $f^i = -Y'^i \in F_0(U)$. Donc α^i est une 1-forme exacte sur U, pour tout i.

Inversement, d'après l'assertion 3.15. de la Proposition 3.15, $D_U = \alpha^i \otimes \frac{\partial}{\partial y^i}$ car $\omega^j = 0$ quel que soit j. Or les α^i sont des formes exactes, alors $\alpha^i = df^i$ où f^i sont des fonctions de $F_0(U)$.

Soit $X = (X^1, \ldots, X^{m+q-p}, X'^1, \ldots, X'^p) \in A_S(U)$, on obtient

$$\begin{aligned}
\alpha^i(X) &= \left(\frac{\partial f^i}{\partial x^1}dx^1 + \frac{\partial f^i}{\partial x^2}dx^2 + \cdots + \frac{\partial f^i}{\partial x^{m+q-p}}dx^{m+q-p}\right)(X) \\
&= X^1\frac{\partial f^i}{\partial x^1} + X^2\frac{\partial f^i}{\partial x^2} + \cdots + X^{m+q-p}\frac{\partial f^i}{\partial x^{m+q-p}}
\end{aligned}$$

Comme

$$D_U\left(X^1, \ldots, X^{m+q-p}, X'^1, \ldots, X'^p\right) = \left(\alpha^i \otimes \frac{\partial}{\partial y^i}\right)(X)$$

$$= \left(X^1\frac{\partial f^i}{\partial x^1} + X^2\frac{\partial f^i}{\partial x^2} + \cdots + X^{m+q-p}\frac{\partial f^i}{\partial x^{m+q-p}}\right)\frac{\partial}{\partial y^i} = -\left[f^i\frac{\partial}{\partial y^i}, X^j\frac{\partial}{\partial x^j}\right]$$

$$= -\left[f^i\frac{\partial}{\partial y^i}, X^j\frac{\partial}{\partial x^j}\right] - \left[f^i\frac{\partial}{\partial y^i}, X'^j\frac{\partial}{\partial y^j}\right] = -\left[f^i\frac{\partial}{\partial y^i}, X^j\frac{\partial}{\partial x^j} + X'^j\frac{\partial}{\partial y^j}\right]$$

car f^i et X'^i ne dépendent pas des y^l.

Alors $D_U^{\alpha,0} = D_U = -\mathrm{L}_{f^i\frac{\partial}{\partial y^i}}$ avec $f^j\frac{\partial}{\partial y^j} \in A_S^2(U)$.

Il en résulte que $D_U^{\alpha,\omega}$ est une dérivation intérieure si et seulement si les $\omega^i \equiv 0$ et $\alpha^i = df^i$, où les f^i sont des fonctions de $F_0(U)$. Dans ce cas, la dérivation $D_U^{\alpha,0} = -\mathrm{L}_{f^i\frac{\partial}{\partial y^i}}$. ■

On rappelle le résultat classique suivant:

Proposition 3.17 *Soit $\mathfrak{A}$ une sous-algèbre de Lie des champs de vecteurs de* M*, Γ un idéal caractéristique de $\mathfrak{A}$, D une dérivation sur $\mathfrak{A}$, π la projection canonique de $\mathfrak{A}$ sur l'algèbre-quotient $\mathfrak{A}/\Gamma$. En posant $D'_\pi(X) = \pi(D(X))$ pour tout $X \in \mathfrak{A}$, D' définit une dérivation sur $\mathfrak{A}/\Gamma$. En particulier, si $D = \mathrm{L}_X$ alors $D' = \mathrm{L}_{\pi(X)}$.*

Proposition 3.18 *Toute dérivation locale D de l'algèbre de Lie A_S s'écrit d'une manière unique sous la forme $\mathrm{L}_X + D^0$ avec $X \in A_S^1$ et, pour toute carte adaptée U, $D^0{}_{|U} = 0$ si la dimension de U est nulle; $D^0{}_{|U} = D^{\alpha,\omega}$ une dérivation définie par la Proposition 3.15 sinon.*

Démonstration. Soit D une dérivation locale de A_S. Il vient que l'algèbre de Lie quotient $A_S(U)/A_S^2(U)$ est isomorphe à $A_S^1(U)$, et est donc isomorphe à l'algèbre de Lie des champs de vecteurs sur un ouvert de $\mathbb{R}^{m+q-p}$. Or toute dérivation de $\chi(\mathbb{R}^{m+q-p})$ est intérieure d'après un résultat de [5], alors toute dérivation de l'algèbre de Lie $A_S(U)/A_S^2(U)$ est intérieure. En vertu de la Proposition 3.17, toute dérivation D_U de $A_S(U)$ est de la forme $D'_U = \mathrm{L}_{\pi(Y)}$, avec $Y \in A_S^1(U)$, où $\pi : A_S(U) \to A_S(U)/A_S^2(U)$ est la projection canonique. En posant $D^0 = D - \mathrm{L}_X$ où $X \in A_S^1$ tel que $X_{|U} = Y$, la dérivation correspondante ${D^0}'_U$ de l'algèbre-quotient est nulle, $D^0{}_U$ est donc une dérivation de $A_S(U)$ dans $A_S^2(U)$. Si $p = 0$ alors $D^0{}_U = 0$. Si $p > 0$, d'après la Proposition 3.15, sur une carte adaptée au feuilletage; $D^0{}_U$ est de la forme $D_U^{\alpha,\omega}$. D'où la décomposition annoncée. ∎

Théorème 3.19 *Si le rang de S est constant égal à $p \in [1, q]$, le premier espace de cohomologie locale de Chevalley-Eilenberg* $\mathrm{H}^1_{loc}(A_S)$ *de* A_S *est isomorphe à* $(\mathrm{H}^1_R(\mathrm{B}) \times \mathbb{R})^p \times \mathbb{R}^{p^2}$*, où* $\mathrm{H}^1_R(\mathrm{B})$ *désigne le premier espace de cohomologie de de Rham sur les formes basiques au feuilletage de* M.

Démonstration. Soit D une dérivation locale de A_S, alors la restriction D_U de D à une carte adaptée U au feuilletage est une dérivation de $A_S(U)$. D'après la Proposition 3.18, D_U se décompose en une somme de deux dérivations $D_U = \mathrm{L}_{X_{|U}} + D_U^{\alpha,\omega}$, où $X_{|U} \in A_S^1(U)$ et, où $D_U^{\alpha,\omega}$ est une dérivation définie dans la Proposition 3.15. Si le rang $p \geq 1$ de S est constant, une dérivation $D_U^{\alpha,\omega}$ s'écrit d'une façon unique $D_U^{\alpha,\omega} = D_U^{\alpha,0} + D_U^{0,\omega}$ et l'expression $D_U^{\alpha,0} \circ D_U^{0,\omega}$ est nulle. L'algèbre des dérivations de la forme $D_U^{0,\omega}$ est isomorphe à l'algèbre $\mathrm{gl}(\mathbb{R}^p)$ des endomorphismes de $A_S^2(U)$. D'autre part, en notant $\alpha_U = (\alpha_U^1, \ldots, \alpha_U^p)$, on a la somme des dérivations $D_U^{\alpha,0} = D_U^{\alpha^1,0} + \cdots + D_U^{\alpha^p,0}$ telles que $D_U^{\alpha^i,0} \circ D_U^{\alpha^j,0} = 0$ pour tous i, j. Les α^i, $i = 1, \ldots, p$ sont des tenseurs invariants par transition des cartes adaptées. L'ensemble des α^i s'identifie à $Z^1(\mathrm{B})_{|U} \times \mathbb{R}$ cf. [4], $Z^1(\mathrm{B})_{|U}$ étant l'ensemble des 1-formes basiques et fermées sur U. L'ouvert U est un domaine d'une carte adaptée quelconque de M, d'où le résultat. ∎

Remarque 3.20 On suppose qu'il existe une feuille singulière du feuilletage $\mathfrak{F}$. En travaillant sur l'ensemble ouvert des points réguliers R dense dans M, on trouve sur la variété R le même résultat que celui de la Proposition 3.18. Si le prolongement de X correspondant à D dans cette proposition est dans A_S^1 et que chaque prolongement de α et de ω sont C^∞, alors la Proposition 3.18 reste valable sur M.

Exemple 3.21 Soit $\mathrm{M} = \mathbb{R}^3$ de coordonnées canoniques (x, y, z) , $S = \left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$. Les éléments de A_S sont de la forme $f(x)\frac{\partial}{\partial x} + g(x)\frac{\partial}{\partial y} + h(x)\frac{\partial}{\partial z}$, pour toutes fonctions C^∞, f, g et h ne dépendant que de x. D'après nos théorèmes, le premier espace de cohomologie de Chevalley-Eilenberg $\mathrm{H}^1(A_S) = \mathrm{H}^1_{loc}(A_S)$ est de dimension six. La Proposition 3.15 donne la construction d'une base des dérivations non intérieures de A_S dont les éléments sont les suivants:

$$D_1 = dy \otimes \frac{\partial}{\partial y} \qquad D_2 = dz \otimes \frac{\partial}{\partial y} \qquad D_3 = dy \otimes \frac{\partial}{\partial z}$$

$$D_4 = dz \otimes \frac{\partial}{\partial z} \qquad D_5 = \psi \otimes \frac{\partial}{\partial y} \qquad D_6 = \psi \otimes \frac{\partial}{\partial z}$$

où ψ désigne l'application $\psi\left(f\left(x\right)\frac{\partial}{\partial x}\right) = \frac{\partial f(x)}{\partial x}$.

Remarque 3.22

1. Si la structure de la variété M feuilletée par $\{X_1, \ldots, X_p\}$ est transversalement orientable, alors chaque forme α^i du Théorème 3.19 s'écrit

$$\alpha^i = \gamma^i + k\varphi$$

où chaque γ^i est une 1-forme basique fermée, k un nombre réel et φ la divergence de la structure transversale.

2. Si C est le champ de Liouville sur le fibré vectoriel TM de la variété M. On désigne par $A_C = \{X \in \mathcal{X}(T\mathrm{M})$ tel que $[X, C] = 0\}$. Soit $\{0\}$ la section nulle de TM, on pose $S = \{C\}$ dans la variété $\overset{\circ}{T}\mathrm{M} = T\mathrm{M} - \{0\}$. L'algèbre de Lie A_S est égale à l'algèbre de Lie $\overset{\circ}{A}_C$ définie dans [2]. Toute dérivation de $\overset{\circ}{A}_C$ est une dérivation indiquée dans la Proposition 3.18. Ce résultat est prolongeable sur TM, d'où le résultat de [4] sur $\mathrm{H}^1(A_C)$.

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Accepted: 19.10.2011

MR3009602 (Review) 17B66 (17B56 53C12)
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Sur les algèbres de Lie d'un système de champs de vecteurs permutables. (French. English, French summaries) [On the Lie algebras of a system of permutable vector fields]
Ital. J. Pure Appl. Math. No. 29 (2012), 163–174.2239-0227

Let M be a smooth paracompact manifold of dimension $m+q$ (where $m,q\geq 1$). Let S be a system of q vector fields on M, mutually commuting and with rank p such that $0\leq p(x)\leq q$ for every $x\in M$. This yields a generalized foliation on M defined by the system S. Denote by A_S the Lie algebra of all vector fields on M that commute with S. The Lie algebra A_S splits into

$$A_S=A_S^1\oplus A_S^2,$$

where A_S^1 is a module over the space of functions that are constant on the leaves and A_S^2 is the Lie algebra generated by S. The main result of the paper under review is to prove that if $0<p(x)<q$ then the following assertions are equivalent:

- Every derivation on A_S is local.
- There exist $X\in A_S^1$ and $h\in F_0(M)$ (functions that are constant on the leaves) such that $X(h)$ doesn't vanish on M.
- $[A_S,A_S]=A_S$.

Furthermore, the authors compute the Chevalley-Eilenberg local cohomology space $\mathrm{H}^1_{\mathrm{loc}}(A_S)$.

Reviewed by *Sofiane Bouarroudj*

African Diaspora Journal of Mathematics
Volume 10, Number 2, pp. 87–95 (2010)
ISSN 1539-854X
www.math-res-pub.org/adjm

SUR LES ALGÈBRES DE LIE DES CHAMPS DE VECTEURS POLYNOMIAUX

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Résumé

On étudie la dérivation de l'algèbre de Lie des champs de vecteurs polynomiaux sur $\mathbb{R}^n$ qui contient tous les champs constants et le champ d'Euler. Elle est adjointe de son normalisateur sur les champs de vecteurs polynomiaux de $\mathbb{R}^n$. Si de plus, l'algèbre de Lie contient tous les champs linéaires diagonaux alors toutes ses dérivations sont intérieures. On donne une classification de cette algèbre de Lie.

AMS Subject Classification : 17B66 ; 17B56 ; 17B40 ; 17B70.

Keywords : Algèbre de Lie, Dérivation, Champs de vecteurs polynomiaux, Champ d'Euler.

1 Introduction

Etant donnée une $\mathbb{R}$-sous-algèbre de Lie $\mathfrak{P}$ de l'algèbre de Lie des champs de vecteurs polynomiaux sur $\mathbb{R}^n$, l'objectif central de notre article est de savoir si toutes les $\mathbb{R}$-dérivations de $\mathfrak{P}$ sont intérieures. Dans l'article [7], l'auteur considère le même problème

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pour l'algèbre de Lie des champs de vecteurs lisses sur une variété différentiable, et montre que toute dérivation est intérieure en utilisant le résultat de [5] sur les operateurs différentiels linéaires. Par un principe analogue, à savoir, une dérivation est une application linéaire et locale, [4] a étudié l'algèbre de Lie des champs de vecteurs analytiques réels et a montré que toute sa dérivation est intérieure en passant par l'étude des dérivations des champs holomorphes sur l'espace de Stein. Or, un champ de vecteurs polynomial nul sur un ouvert est trivial, donc la condition locale de sa dérivation ne marche pas. [2] a utilisé le " toral method " pour étudier les dérivations de certaines algèbres de Lie (nilpotentes) en tenant compte du calcul de leurs dimensions. Dans ce papier, nous étudions les $\mathbb{R}$-sous-algèbres de Lie $\mathfrak{P}$ des champs de vecteurs polynomiaux sur $\mathbb{R}^n$ qui contiennent tous les champs constants et le champ d'Euler. On démontre que le normalisateur $\mathfrak{N}$ de $\mathfrak{P}$ est une sous-algèbre de Lie des champs de vecteurs polynomiaux de $\mathbb{R}^n$. En utilisant une graduation de $\mathfrak{P}$, et le rôle des champs constants et du champ d'Euler par le crochet des champs de vecteurs, on trouve que toute sa dérivation est adjointe par rapport à son normalisateur. On montre de plus que si tous les champs linéaires diagonaux appartiennent à $\mathfrak{P}$ alors toute sa dérivation est intérieure. En particulier, le normalisateur $\mathfrak{N}$ de $\mathfrak{P}$ contient tous les champs linéaires diagonaux. Le centralisateur de $\mathfrak{P}$ est réduit à zéro. Ainsi, on peut calculer le premier espace de cohomologie de Chevalley-Eilenberg de $\mathfrak{P}$ et de $\mathfrak{N}$. On donne des exemples d'illustration de ces résultats, des exemples d'algèbres de Lie de champs de vecteurs polynomiaux qui ne vérifient pas la condition sus-mentionnée, et où il existe une dérivation ne provenant pas des champs de vecteurs de $\chi(\mathbb{R}^n)$. La classification des algèbres de Lie simples complexes (et réels) est bien connue. C'est en partie dûe aux travaux d'Elie Cartan, de Dynkin et de Killing. Ici, en utilisant la divergence de champs de vecteurs sur $\mathbb{R}^n$, on peut classifier les algèbres de Lie $\mathfrak{P}$ contenant tous les champs constants et le champ d'Euler. Le calcul du premier espace de cohomologie de Chevalley-Eilenberg des sous-algèbres de l'algèbre de Lie des dérivations d'un anneau associatif, commutatif et unitaire ; a été traité dans [6]. Mais en fait, les hypothèses dans [6] p.71 ne permettent pas d'aboutir à tous nos résultats.

2 Etude des algèbres de Lie de $\mathfrak{P}$

On désigne par $\chi(\mathbb{R}^n)$ l'algèbre de Lie des champs de vecteurs sur $\mathbb{R}^n$. Dans tout ce paragraphe, on considère une $\mathbb{R}$-sous-algèbre de Lie $\mathfrak{P}$ (consistant) de champs de vecteurs polynomiaux sur $\mathbb{R}^n$ qui contient tous les champs constants et le champ d'Euler E , où $E = \sum_i x^i \frac{\partial}{\partial x^i}$ en coordonnées $\left(x^i\right)_{1\leq i\leq n}$ de $\mathbb{R}^n$, $\mathfrak{N}$ son normalisateur dans $\chi(\mathbb{R}^n)$; H_i l'espace vectoriel des champs homogènes de degré i avec $i \in \mathbb{N}\cup\{-1\}$. On note H_0^d l'algèbre de Lie engendrée par tous les champs linéaires diagonaux. Les champs constants sont homogènes de degré -1 et les champs linéaires de degré 0. On désigne par L_X la dérivée de Lie par rapport à $X \in \chi(\mathbb{R}^n)$. On adopte la convention d'Einstein sur la sommation d'indices sauf mention expresse.

Le crochet de deux champs de vecteurs $X = X^i \frac{\partial}{\partial x^i}$ et $Y = Y^j \frac{\partial}{\partial x^j}$ de $\chi(\mathbb{R}^n)$ en coordonnées $\left(x^i\right)_{1\leq i\leq n}$ est donné par :

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j} \tag{2.1}$$

En utilisant la formule (2.1), $\mathfrak{P}$ est graduée de la façon suivante :
$\mathfrak{P} = \bigoplus_{i\geq -1} \mathfrak{P}_i$, où chaque $\mathfrak{P}_i$ est un sous-espace vectoriel de dimension finie de H_i, tels que

$$[\mathfrak{P}_{-1}, \mathfrak{P}_{-1}] = \{0\} \text{ et } \forall i, j \geq -1 \text{ , où } i+j \geq -1,\ [\mathfrak{P}_i, \mathfrak{P}_j] \subset \mathfrak{P}_{i+j} \tag{2.2}$$

Proposition 2.1. *Un champ X de $\mathfrak{P}$ est homogène de degré p si et seulement si $[E,X] = pX$.*

Démonstration. D'après (2.1) et (2.2), on a le résultat. □

Remarque 2.2. On peut identifier un champ linéaire c'est à dire de degré 0 à un et un seul élément de $gl(n,\mathbb{R})$, cf.[3] p.5.

Lemme 2.3. *Si $F = \sum_{-1\leq i\leq k} F_i \in \mathfrak{P}$ suivant la graduation de $\mathfrak{P}$, alors $\forall i,\ F_i \in \mathfrak{P}$.*

Démonstration. Soit $F = \sum_{-1\leq i\leq k} F_i$, la décomposition d'un champ de vecteurs F de $\mathfrak{P}$ en ses composantes homogènes.
$[F,E] = \sum_{-1\leq i\leq k} (-i) F_i \in \mathfrak{P}$, on peut recommencer $(k-1)$ fois ce processus et on a :
$\forall j \in [0,k],\ \sum_{-1\leq i\leq k} (-i)^j F_i \in \mathfrak{P}$, d'où un système linéaire à la Vandermonde dont on déduit que tous les F_i appartiennent à $\mathfrak{P}$. □

Définition 2.4. Le normalisateur $\mathfrak{N}$ de $\mathfrak{P}$ est défini par $\mathfrak{N} = \{X \in \chi(\mathbb{R}^n) \,/\, [X,\mathfrak{P}] \subset \mathfrak{P}\}$.

Proposition 2.5. *Le normalisateur $\mathfrak{N}$ est une $\mathbb{R}$-algèbre de Lie (consistant) des champs de vecteurs polynomiaux sur $\mathbb{R}^n$.*

Démonstration. $\mathfrak{N}$ étant une $\mathbb{R}$-algèbre de Lie cf. [1].
Soit $X^i \frac{\partial}{\partial x^i} \in \mathfrak{N}$ dans $\chi(\mathbb{R}^n)$, où $(x^j)_{1\leq j\leq n}$ est un système de coordonnées de $\mathbb{R}^n$.
Pour tout j, $\left[X^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = -\frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i} \in \mathfrak{P}$ alors chaque $\frac{\partial X^i}{\partial x^j}$ est un polynôme.

Donc $\left[X^i \frac{\partial}{\partial x^i}, x^k \frac{\partial}{\partial x^k}\right] = \left(X^k - x^i \frac{\partial X^k}{\partial x^i}\right)\frac{\partial}{\partial x^k}$ est un champ de vecteurs polynomial.
D'où $\forall k$, X^k sont des polynômes. □

Définition 2.6. Une dérivation D de $\mathfrak{P}$ est une application $\mathbb{R}-$linéaire de $\mathfrak{P}$ dans $\mathfrak{P}$ telle que $\forall X,Y \in \mathfrak{P},\ D[X,Y] = [D(X),Y] + [X,D(Y)]$.

Proposition 2.7. *Une dérivation D de $\mathfrak{P}$ est nulle sur tous les champs constants et le champ d'Euler si et seulement si D est nulle.*

Démonstration. Supposons que $D(E) = D(C) = 0$, où $C \in H_{-1}$.
Compte tenu du Lemme 2.3, soit $P \in H_k$, où $k \in \mathbb{N}\cup\{-1\}$ et raisonnons par récurrence.
Pour $P \in H_{-1}$, on a par hypothèse $D(P) = 0$.
On suppose que $D(P) = 0$ soit vrai jusqu'au rang $k > -1$.
Si $P \in H_{k+1}$, d'après (2.1) on a $D[C,P] = (k+1)D(Q)$, où $Q \in H_k$. Donc c'est nul par hypothèse.
Par la définition d'une dérivation, $[D(C),P] + [C,D(P)] = 0$.
C'est à dire pour tout $C \in H_{-1}$, $[C,D(P)] = 0$, ainsi $D(P) \in H_{-1}$.

On pose $D(P) = C'$, d'après la Proposition 2.1 on a $D[E,P] = kD(P)$.
Par définition et par hypothèse sur D, $kC' = [E,C'] = -C'$, ainsi $C' = 0$.
D'où $\forall k \geq 0,\ D(P) = 0$ et $D \equiv 0$.
La réciproque est évidente. □

Proposition 2.8. *Si D est une dérivation homogène de degré 0 sur $\mathfrak{P}$, nulle sur E alors $D = L_X$ avec X un champ de vecteurs polynomial homogène de degré 0.*

Démonstration. Soit D une dérivation homogène de degré 0 de $\mathfrak{P}$, c'est à dire pour tout $i \in \mathbb{N} \cup \{-1\}$, $D(H_i \cap \mathfrak{P}) \subset H_i \cap \mathfrak{P}$. Alors pour $C \in H_{-1},\ D(C) = C'$, où $C' \in H_{-1}$.
Par (2.1) $\exists ! X \in H_0$ tel que $D(C) = [X,C]$ pour tout $C \in H_{-1}$ et $D(E) = 0 = [X,E]$.
Donc $D_{|H_{-1}\oplus\langle E\rangle} = L_X$ et d'après la Proposition 2.7, $D = L_X$. □

Théorème 2.9. *Toute dérivation de $\mathfrak{P}$ est intérieure par rapport au normalisateur $\mathfrak{N}$ de $\mathfrak{P}$. De plus $D = L_{(F+X)}$ avec $X \in H_0$ et $F \in \mathfrak{P} - H_0$.*

Démonstration. Soit D une dérivation de $\mathfrak{P}$. D'après le Lemme 2.3, il suffit de considérer $D(V_m)$, où V_m est un champ homogène de degré $m \geq -1$ de $\mathfrak{P}$.
On pose $D(V_m) = \sum\limits_{-1 \leq i \leq k} W_i$ et $D(E) = \sum\limits_{-1 \leq i \leq l} E_i$, où les $W_i,\ E_i \in H_i \cap \mathfrak{P}$ en vertu du Lemme 2.3.
Utilisant la définition de la dérivation en terme de composantes homogènes de même degré, on a :

$$D[E,V_m] = [D(E),V_m] + [E,D(V_m)]$$

Entraîne

$$\begin{array}{ccccc}
mW_{-1} & = & 0 & - & W_{-1} \\
mW_0 & = & 0 & + & 0 \\
mW_1 & = & 0 & + & W_1 \\
mW_2 & = & 0 & + & 2W_2 \\
\dots & = & \dots & & \dots \\
mW_{m-2} & = & 0 & + & (m-2)W_{m-2} \\
mW_{m-1} & = & [E_{-1},V_m] & + & (m-1)W_{m-1} \\
mW_m & = & [E_0,V_m] & + & mW_m \\
\dots & = & \dots & & \dots \\
mW_{m+l} & = & [E_l,V_m] & + & (m+l)W_{m+l} \\
mW_{m+l+1} & = & 0 & + & (m+l+1)W_{m+l+1} \\
mW_{m+l+2} & = & 0 & + & (m+l+2)W_{m+l+2} \\
\dots & = & \dots & & \dots
\end{array}$$

- Pour $-1 \leq i \leq m-2$ et $i > m+l$, on a $(i-m)W_i = 0 \Rightarrow W_i = 0$
- Pour $m-1 \leq i \leq m+l$, on obtient $(i-m)W_i + [E_{i-m},V_m] = 0$.
  En particulier

$$[E_0,V_m] = 0 \tag{2.3}$$

Par suite,

$$
\begin{aligned}
D(V_m) &= [E_{-1},V_m]+W_m+\sum_{m+1\le i\le m+l}\left(\frac{1}{m-i}\right)[E_{i-m},V_m]\\
&= [E_{-1},V_m]+W_m+\sum_{1\le i\le l}\left(\frac{-1}{i}\right)[E_i,V_m]\\
&= W_m+\left[\sum_{i\neq 0}\left(\frac{-1}{i}\right)E_i,V_m\right]
\end{aligned}
$$

On note $F=\sum_{-1\le i\le l, i\neq 0}\left(\frac{-1}{i}\right)E_i\in\mathfrak{P}$, et on a $D(V_m)=W_m+[F,V_m]$.
Posant $D'=D-L_F$, c'est une dérivation homogène de degré 0 de $\mathfrak{P}$.
D'après (2.3) et le fait que tous les champs constants sont dans $\mathfrak{P}$, on a $[E_0,C]=0\;\forall C\in H_{-1}$.
Par conséquent, $E_0=0$ et $D'(E)=0$.
D'après la Proposition 2.8, $D'=L_X$ avec $X\in H_0$.
Alors $D=L_{(F+X)}$ avec $F+X$ est un champ de vecteurs polynomial sur $\mathbb{R}^n$.
Dans ce cas, $F+X\in\mathfrak{N}$ avec $F\in\mathfrak{P}-H_0$. □

Remarque 2.10. Le champ $F+X$ du Théorème 2.9 peut ne pas être dans $\mathfrak{P}$, car L_X avec X un champ linéaire diagonal est toujours une dérivation de $\mathfrak{P}$. Or, tous les champs linéaires diagonaux ne sont pas forcément dans $\mathfrak{P}$.

Proposition 2.11. *Le centralisateur $\mathfrak{C}(\mathfrak{P})$ de $\mathfrak{P}$ est nul.*

Démonstration. Par définition $\mathfrak{C}(\mathfrak{P})=\{X\in\chi(\mathbb{R}^n)\,/\,[X,\mathfrak{P}]=\{0\}\}$.
Soit $X\in\mathfrak{C}(\mathfrak{P})\subset\mathfrak{N}$, X est un champ de vecteurs polynomial d'après la Proposition 2.5.
Alors $X=\sum_{-1\le i\le k}X_i$ avec $X_i\in H_i$.
De la Proposition 2.1 et par définition, $[E,X]=\sum_{-1\le i\le k}(i)X_i=0$. Alors $\forall i\neq 0$ on a $X_i=0$.
Or $[C,X]=[C,X_0]=0\;\forall C\in H_{-1}$, alors $X_0=0$ et $X=0$.
D'où $\mathfrak{C}(\mathfrak{P})=\{0\}$. □

Théorème 2.12. *Le premier espace de cohomologie de Chevalley-Eilenberg de $\mathfrak{P}$ noté $\mathrm{H}^1(\mathfrak{P})$ est isomorphe à une $\mathbb{R}$-sous-algèbre de Lie de H_0. Si tous les champs linéaires diagonaux appartiennent à $\mathfrak{P}$ alors $\mathrm{H}^1(\mathfrak{P})$ est nul.*

Démonstration. Par définition, $\mathrm{H}^1(\mathfrak{P})=\mathrm{Der}(\mathfrak{P})/\mathrm{ad}_{\mathfrak{P}}$ cf. [8] avec $\mathrm{Der}(\mathfrak{P})$ l'algèbre de Lie des dérivations de $\mathfrak{P}$ et $\mathrm{ad}_{\mathfrak{P}}$ l'ensemble des dérivations intérieures de $\mathfrak{P}$.
Soit $L:X\in\mathfrak{N}\longmapsto\mathrm{ad}_X\in\mathrm{Der}(\mathfrak{P})$ qui est $\mathbb{R}$-linéaire.
Si $L_X=0$, c'est à dire $[X,Y]=0\;\forall Y\in\mathfrak{P}$, alors $X\in\mathfrak{C}(\mathfrak{P})$.
Par la Proposition 2.11, $\mathfrak{C}(\mathfrak{P})=\{0\}$ et $X=0$. D'où l'injectivité de L.
Soit $D\in\mathrm{Der}(\mathfrak{P})$, d'après le Théorème 2.9 $\exists X\in\mathfrak{N}$ tel que $D=L_X$. D'où la surjectivité.
Par ailleurs, pour $X,Y,Z\in\chi(\mathbb{R}^n)$ $[X,[Y,Z]]+[[X,Z],Y]=[X,[Y,Z]]-[Y,[X,Z]]$ et en utilisant l'identité de Jacobi, on a $[[X,Y],Z]=L_XL_Y(Z)-L_YL_X(Z)$.
Donc $L_{[X,Y]}(Z)=[L_X,L_Y](Z)\;\forall X,Y\in\mathfrak{N}$ et $\forall Z\in\chi(\mathbb{R}^n)$, ainsi L est un homomorphisme d'algèbres de Lie.
Par suite $\mathfrak{N}\cong\mathrm{Der}(\mathfrak{P})$.

D'une manière analogue que précédemment, $L' : X \in \mathfrak{P} \longmapsto \mathrm{ad}_X \in \mathrm{ad}_{\mathfrak{P}}$ est $\mathbb{R}$-linéaire. C'est donc un isomorphisme d'algèbres de Lie. Par suite $\mathfrak{P} \cong \mathrm{ad}_{\mathfrak{P}}$.
Alors

$$\mathrm{H}^1(\mathfrak{P}) \cong \mathfrak{N}/\mathfrak{P}$$

, où $\mathfrak{N}/\mathfrak{P}$ est isomorphe à une $\mathbb{R}$-sous-algèbre de Lie de H_0 d'après le Théorème 2.9.
Par ailleurs, soit $D = L_{(F+X)}$ avec $F \in \mathfrak{P}$, $X \in H_0$ du Théorème 2.9.
$\exists Y \in \left(H_0 - H_0^d\right) \cup \{0\}$ et $C_0 \in H_0^d$ tels que $X = C_0 + Y$.
Par (2.1), il existe $C_0' \in H_0^d$ tel que $[Y, C_0'] = Y$. Si $H_0^d \subset \mathfrak{P}$ alors $Y \in \mathfrak{P}$ car $L_Y \in \mathrm{Der}(\mathfrak{P})$.
Ainsi $X \in \mathfrak{P}$ et toute dérivation de $\mathfrak{P}$ est intérieure. □

Corollaire 2.13. *Toute dérivation de $\mathfrak{N}$ est intérieure .*

Démonstration. D'après la *Remarque* 2.10, le normalisateur $\mathfrak{N}$ contient tous les champs linéaires diagonaux. Alors par le Théorème 2.12 on a le résultat. □

Exemple 2.14. L'algèbre de Lie des champs affines $H_{-1} \oplus H_0$ sur $\mathbb{R}^n$ de dimension $n^2 + n$ est résoluble d'ordre 3, toute sa dérivation est intérieure par le Corollaire 2.13.

Exemple 2.15. Chacune des $\mathbb{R}$-algèbres de Lie suivantes est un exemple de $\mathfrak{P}$.

1. Dans $\mathbb{R}^2$, l'algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ est résoluble d'ordre 2 et sa dérivation D est L_X avec X appartient à l'algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}$, $, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}$. Ainsi son normalisateur est $H_{-1} \oplus H_0$ et $\mathrm{H}^1(\mathfrak{P}) \cong H_0/ < E >$.
2. Sur $\mathbb{R}^2$, soit l'algèbre de Lie non simple et non résoluble, engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, $x\frac{\partial}{\partial x}$, $, y\frac{\partial}{\partial y}, (x)^2\frac{\partial}{\partial x}$. Sa dérivation D est L_X telle que X appartenant à l'algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, (x)^2\frac{\partial}{\partial x}$, et $\mathrm{H}^1(\mathfrak{P}) \cong \{0\}$.
3. Sur $\mathbb{R}^2$, on considère l'algèbre de Lie résoluble d'ordre 3, engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} +$ $+ y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, (x)^2\frac{\partial}{\partial y}$. Sa dérivation D est L_X avec X appartient à l'algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, (x)^2\frac{\partial}{\partial y}$, et $\mathrm{H}^1(\mathfrak{P}) \cong H_0^d/ < E >$.
4. Sur $\mathbb{R}^2$, l'algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, (x)^3\frac{\partial}{\partial x}$ est de dimension infinie dénombrable, dont le seul idéal commutatif est $\{0\}$, et $[\mathfrak{P}, \mathfrak{P}] = \mathfrak{P}$. Sa dérivation D est L_X telle que X appartenant à l'algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, (x)^3\frac{\partial}{\partial x}$, et $\mathrm{H}^1(\mathfrak{P}) \cong \{0\}$.
5. On considère l'algèbre de Lie sur $\mathbb{R}^2$ de dimension infinie dénombrable, engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, xy\frac{\partial}{\partial x}$. Elle a un idéal commutatif non nul engendré par $\left\{ \underset{r \geq 0}{y^r \frac{\partial}{\partial x}} \right\}$ et coïncide avec son idéal dérivé. Sa dérivation D est L_X avec X appartient à l'algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, xy\frac{\partial}{\partial x}$, et $\mathrm{H}^1(\mathfrak{P}) \cong \{0\}$.
6. Sur $\mathbb{R}^2$, soit l'algèbre de Lie de dimension infinie dénombrable engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, $,x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, (x)^2\frac{\partial}{\partial x}$. Sa dérivation D est L_X telle que X appartenant à l'algèbre de Lie engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x}$. $\mathrm{H}^1(\mathfrak{P}) \cong \{0\}$ et $[\mathfrak{P}, \mathfrak{P}] \underset{\neq}{\subset} \mathfrak{P}$.

Remarque 2.16. Si un champ constant ou le champ d'Euler E n'appartient pas à $\mathfrak{P}$ alors il peut exister des dérivations qui ne sont pas intérieures dans le normalisateur de $\mathfrak{P}$:
Soit la $\mathbb{R}$-algèbre de Lie des champs de vecteurs polynomiaux engendrée par $\frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}$ sur $\mathbb{R}^2$. Sa dérivation définie par $D\left(\frac{\partial}{\partial x}\right) = D\left(y\frac{\partial}{\partial y}\right) = 0,\ D\left(x\frac{\partial}{\partial x}\right) = y\frac{\partial}{\partial y}$ est une dérivation non intérieure de $\chi\left(\mathbb{R}^2\right)$.
La $\mathbb{R}$-algèbre de Lie dans $\mathbb{R}^2$ engendrée par $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y}$ (resp. $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}$) admet une dérivation D (resp. D') non intérieure de son normalisateur, définie respectivement par :

$$D\left(\frac{\partial}{\partial x}\right) = D\left(\frac{\partial}{\partial y}\right) = 0,\ D\left(y\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}$$

$$D'\left(\frac{\partial}{\partial x}\right) = D'\left(\frac{\partial}{\partial y}\right) = 0,\ D'\left(x\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}$$

Définition 2.17. On désigne par λ_X le degré d'homogénéité d'un champ homogène X de $\mathfrak{P}$. $\lambda_{\mathfrak{S}}$ est le degré d'homogénéité maximal de tous les champs homogènes de $\mathfrak{S} \subset \mathfrak{P}$. Ce dernier peut être infini.

Lemme 2.18. *Soient X un champ homogène de degré $k \geq 1$, $(V_i)_i$ une suite de sous-espace de $\sum_{-1 \leq i \leq k} H_i$ définie par $V_0 = \mathbb{R}X$ et la relation de récurrence : pour tout $i \in \mathbb{N}$, $V_{i+1} = [H_{-1} \oplus \langle E \rangle, V_i]$. Cette suite stationne en V_{k+1}. Si* $\operatorname{div}(X) = 0$ *, où* div *est la divergence de champ de vecteurs de $\chi(\mathbb{R}^n)$ alors $\forall Y \in V_k,\ \operatorname{div}(Y) = 0$.*

Démonstration. Soit X un champ homogène de degré $k \geq 1$. En vertu du rôle des champs constants et du champ d'Euler par le crochet de champs de vecteurs, la suite $(V_i)_i$ stationne lorsque $\exists j$ et $Y \in V_j$ tels que $\lambda_Y = -1$. Dans ce cas $(V_i)_i$ converge vers V_{k+1}.
La formule de divergence donne $\operatorname{div}([A,B]) = A(\operatorname{div}(B)) - B(\operatorname{div}(A))$ et $\operatorname{div}([E,A]) = E\operatorname{div}(A)\ \forall A, B \in \chi(\mathbb{R}^n)$. Par conséquent, si $\operatorname{div}(X) = 0$ alors $\operatorname{div}(Y) = 0,\ \forall Y \in V_{k+1}$. □

Lemme 2.19. *Il existe un champ quadratique ω de $\mathfrak{P}$ de divergence non nulle tel que $\left[\omega, \mathfrak{P} - \left(H_{-1} \oplus H_0^d\right)\right] \neq \{0\}$ si et seulement si (il existe un champ quadratique Y de $\mathfrak{P}$ de divergence non nulle et $\exists Z \in \left(H_0 - H_0^d\right) \cap \mathfrak{P}$) ou (il existe un champ quadratique Y de $\mathfrak{P}$ de divergence non nulle et $\exists Z \in H_2 \cap \mathfrak{P}$) tel que le degré $(\lambda_{V_i})_i$ tend vers $+\infty$, où $(V_i)_i$ la suite définie par $V_0 = Y$ et la relation de récurrence $\forall i \in \mathbb{N},\ V_{i+1} = [Z, V_i]$.*

Démonstration. S'il existe un champ quadratique ω de $\mathfrak{P}$ avec $\operatorname{div}(\omega) \neq 0$ tel que

$$\left[\omega, \mathfrak{P} - \left(H_{-1} \oplus H_0^d\right)\right] \neq \{0\}$$

Comme $\operatorname{div}(\omega) \neq 0$, alors $\exists C \in H_{-1}$ tel que $[C, \omega] \neq 0$.
1^er cas : Il existe $C' \in H_{-1}$ avec $[C', \omega] \in \left(H_0 - H_0^d\right) \cap \mathfrak{P}$ tel que le degré de la suite $(V_i)_i$ tend vers $+\infty$, où $(V_i)_i$ est définie par $V_0 = \omega$ et la récurrence $V_{i+1} = [[C', \omega], V_i]$.
2^ème cas : $\exists C' \in H_{-1}$ tel que $[C', \omega] \in H_0^d \cap \mathfrak{P}$ et $\forall h \in H_{-1}$, $[h, \omega] \notin \left(H_0 - H_0^d\right)$.
Soit $X \in \mathfrak{P} - \left(H_{-1} \oplus H_0^d\right)$, avec $\deg(X) = k \geq 1$ tel que $[\omega, X] \neq 0$.

1. Pour $k = 1$: Il existe $C'' \in H_{-1}$ avec $[C'', X] \in \left(H_0 - H_0^d\right) \cap \mathfrak{P}$ tel que le degré de la suite $(V_i)_i$ tend vers $+\infty$, où $(V_i)_i$ est définie par $V_0 = \omega$ et la récurrence $V_{i+1} = [[C'', X], V_i]$.

2. Pour $k \geq 2$: Des crochets successifs avec les champs constants permettent de ramener X au degré 2.
   - Si $\exists C''' \in H_{-1}$ avec $[C''',X] = \alpha\omega + \omega' \in H_1 \cap \mathfrak{P}$, où $\alpha \in \mathbb{R}^*$ alors le degré de la suite $(V_i)_i$ tend vers $+\infty$, où $(V_i)_i$ est définie par $V_0 = \omega$ et la récurrence $V_{i+1} = [X,V_i]$.
   - Si $\forall C''' \in H_{-1},\ [C''',X] \neq \alpha\omega + \omega' \in H_1 \cap \mathfrak{P}$ alors on peut faire des crochets successifs de X avec les champs constants jusqu'à ce qu'on obtienne le sous-cas 1..

La réciproque est évidente. □

Proposition 2.20. *Il existe un champ quadratique ω de $\mathfrak{P}$ de divergence non nulle tel que $\left[\omega, \mathfrak{P} - \left(H_{-1} \oplus H_0^d\right)\right] \neq \{0\}$ si et seulement si $\lambda_{\mathfrak{P}}$ est infini.*

Démonstration. D'après le Lemme 2.19, on a la nécessité.
Réciproquement, si pour tout $\omega \in H_1 \cap \mathfrak{P}$ tel que $\operatorname{div}(\omega) \neq 0$ et $\left[\omega, \mathfrak{P} - \left(H_{-1} \oplus H_0^d\right)\right] = \{0\}$, montrons que $\lambda_{\mathfrak{P}}$ est fini.
Soit alors ω un champ vérifiant ces hypothèses. On peut décomposer ω en une somme des champs quadratiques $\omega = \sum\limits_i \omega_i$ tels que $\Delta\omega_i \neq 0\ \forall i$, où Δ désigne le laplacien de champ de vecteurs de $\chi(\mathbb{R}^n)$, sinon il existe $C \in H_{-1}$ tel que $[C,\omega] \in \left(H_0 - H_0^d\right) \cap \mathfrak{P}$ et $[\omega,[C,\omega]] \neq 0$.
Alors les champs de vecteurs polynomiaux homogènes X de degré supérieur ou égal à 2 de $\mathfrak{P}$ sont de divergence nulle avec $[\omega,X] = 0$, et les champs homogènes Y de degré inférieur ou égal à 1 de $\mathfrak{P} - \left(H_{-1} \oplus H_0^d\right)$ vérifient $[\omega,Y] = 0$.
Donc le degré maximal des champs de $\mathfrak{P}$ dépend de l'existence dans $\mathfrak{P}$ d'un champ homogène de degré arbitraire supérieur ou égal à 2.
D'où $\lambda_{\mathfrak{P}} \geq 2$ est fini. □

Théorème 2.21. *Les seules $\mathbb{R}$-algèbres de Lie des champs de vecteurs polynomiaux $\mathfrak{P}$ sur $\mathbb{R}^n$ contenant tous les champs constants et le champ d'Euler sont :*

1. *Les $\mathbb{R}$-sous-algèbres de Lie de $H_{-1} \oplus H_0$ contenant tous les champs constants et le champ d'Euler.*
2. *Les $\mathbb{R}$-algèbres de Lie de champs de vecteurs polynomiaux sur $\mathbb{R}^n$ contenant tous les champs constants et le champ d'Euler, dont il existe un champ homogène de divergence non nulle de H_k avec $k \geq 1$ vérifiant : $\lambda_{\mathfrak{P}}$ est infini si et seulement s'il existe un champ quadratique ω de $\mathfrak{P}$ de divergence non nulle tel que $\left[\omega, \mathfrak{P} - \left(H_{-1} \oplus H_0^d\right)\right] \neq \{0\}$.*
3. *Les $\mathbb{R}$-algèbres de Lie des champs de vecteurs polynomiaux sur $\mathbb{R}^n$ contenant tous les champs constants et le champ d'Euler, dont les champs homogènes de degré supérieur ou égal 1 sont de divergence nulle ($\lambda_{\mathfrak{P}}$ peut être fini ou infini).*

Démonstration.
1. Immédiate.
   Sinon $\lambda_{\mathfrak{P}} \geq 1$, car $\mathfrak{P}$ n'est plus un sous-espace de $H_{-1} \oplus H_0$.
2. S'il existe $X \in H_k \cap \mathfrak{P}$, où $k \geq 1$ tel que $\operatorname{div}(X) \neq 0$, d'après le Lemme 2.18 et la Proposition 2.20, on peut construire toutes les $\mathbb{R}$-algèbres de Lie des champs de vecteurs polynomiaux satisfaisant :
   $\lambda_{\mathfrak{P}}$ est infini si et seulement s'il existe un champ $\omega \in H_1 \cap \mathfrak{P}$, avec $\operatorname{div}(\omega) \neq 0$ tel que $\left[\omega, \mathfrak{P} - \left(H_{-1} \oplus H_0^d\right)\right] \neq \{0\}$.

3. Si $\forall X \in H_k \cap \mathfrak{P}$, où $k \geq 1$ tel que $\operatorname{div}(X) = 0$, en vertu du Lemme 2.18, on peut construire toutes les $\mathbb{R}$-algèbres de Lie des champs de vecteurs polynomiaux vérifiant :
$\forall Y \in \mathfrak{P} - H_0^d,\ \operatorname{div}(Y) = 0$ et $\forall Y, Z \in \mathfrak{P}, \operatorname{div}([Y,Z]) = 0$.

$\square$

Corollaire 2.22. *En coordonnées* $\left(x^i\right)_{1\leq i\leq n}$*de* $\mathbb{R}^n$*, la valeur de* $\lambda_{\mathfrak{P}}$ *est infini si et seulement si* $\left(\exists i \neq j \ / \ x^i x^j \frac{\partial}{\partial x^j}\right)$ *ou* $\left(\exists i \neq j \ / \ x^j \frac{\partial}{\partial x^i}, \left(x^j\right)^2 \frac{\partial}{\partial x^j}\right)$ *ou* $\left(\exists i \ / \ \left(x^i\right)^3 \frac{\partial}{\partial x^i}\right)$ *figurant dans l'expression d'un élément non nul de* $\mathfrak{P}$.

Proposition 2.23. *Toute dérivation des champs de vecteurs polynomiaux* $\mathfrak{P}$ *sur* $\mathbb{R}$ *est intérieure, toute dérivation des champs de vecteurs polynomiaux* $\mathfrak{P}$ *de dimension infinie sur* $\mathbb{R}^2$ *est intérieure.*

Démonstration. C'est une conséquence du Théorème 2.12 et du Théorème 2.21. $\square$

Nous tenons à remercier M. Jean Moulin Ollagnier, Université Paris XII ; M. Didier Pinchon, Université Toulouse III ; pour leurs suggestions au cours de ce travail.

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MR2774261 (2012b:17048) 17B66 (17B40)
Ravelonirina, H. S. G. (MAD-UANTS-MIF);
Randriambololondrantomalala, P. (MAD-UANTS-MIF); **Anona, M.** (MAD-UANTS-MIF)
Sur les algèbres de Lie des champs de vecteurs polynomiaux. (French. French summary)
[On the Lie algebras of polynomial vector fields]
Afr. Diaspora J. Math. **10** (2010), *no. 2,* 87–95.

The authors consider a Lie subalgebra $\mathfrak{B}$ of $\mathfrak{X}(\mathbb{R}^n)$, the Lie algebra of vector fields on $\mathbb{R}^n$, where $\mathfrak{B}$ consists of polynomial vector fields, and they are interested in conditions under which every derivation of $\mathfrak{B}$ is inner. The minimum condition imposed on $\mathfrak{B}$ throughout the paper is that it contain all constant vector fields as well as the Euler vector field $E=\sum_i x^i\frac{\partial}{\partial x^i}$, where $(x^i)_{1\le i\le n}$ is a coordinate system in $\mathbb{R}^n$. It is then shown that the normalizer $\mathfrak{N}$ of $\mathfrak{B}$ also consists of polynomial vector fields, and that every derivation of $\mathfrak{B}$ is the adjoint of some element of $\mathfrak{N}$. If in addition it is assumed that all linear diagonal vector fields are in $\mathfrak{B}$, then the first Chevalley-Eilenberg cohomology space of $\mathfrak{B}$ reduces to zero, and thus every derivation of $\mathfrak{B}$ is inner. A classification of all subalgebras $\mathfrak{B}$ of $\mathfrak{X}(\mathbb{R}^n)$ containing all constant vector fields as well as the Euler vector field is given, based on the divergence of homogeneous vector fields. Several simple examples of subalgebras $\mathfrak{B}$ are also given, some of which show that if $\mathfrak{B}$ does not contain a constant vector field or Euler vector field, then it might have a derivation which is not inner with respect to its normalizer $\mathfrak{N}$.

Reviewed by *Jean-Claude Ndogmo*

African Diaspora Journal of Mathematics
Volume 10, Number 2, pp. 135–144 (2010)

ISSN 1539-854X
www.math-res-pub.org/adjm

SUR LES ALGÈBRES DE LIE D'UNE DISTRIBUTION ET D'UN FEUILLETAGE GÉNÉRALISÉ

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Résumé

On étudie le premier espace de cohomologie de Chevalley-Eilenberg des algèbres de Lie attachées à une distribution non régulière involutive d'une variété différentiable. On applique les résultats obtenus à l'algèbre de Lie des champs de vecteurs à support compact et aux algèbres de Lie relatives à un feuilletage généralisé, conduisant à une généralisation d'un théorème de Kanie et de certains résultats de Lichnérowicz.

AMS Subject Classification : Primary 17B66, Secondary 17B56, 53C12.

Keywords : Algèbre de Lie, Distribution non-régulière, Cohomologie de Chevalley-Eilenberg, Feuilletage généralisé, Champs de vecteurs à support compact.

1 Introduction

En 1960, Peetre cf. [7] a prouvé que tout opérateur différentiel linéaire sur l'anneau des fonctions réelles d'une variété différentiable s'écrit localement en une somme finie de dérivations sur cet anneau. En utilisant ces résultats, Takens cf. [8] a démontré que toute dérivation de l'algèbre de Lie des champs de vecteurs sur une variété différentiable est

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intérieure. Cette étude a été étendue par Kanie cf. [4] et Lichnérowicz cf. [6] dans le cas des algèbres de Lie attachées à un feuilletage régulier. Soient M une variété différentiable, $F(M)$ l'anneau des fonctions réelles de M, $\chi(M)$ l'algèbre de Lie des champs de vecteurs sur M. Dans ce papier, on se propose d'étudier la dérivation d'une distribution involutive Ω de classe C^∞ sur M. L'algèbre de Lie Ω vérifie, pour tout $x \in M$ il existe $X \in \Omega$ tel que $X(x) \neq 0$. On montre que toute dérivation de Ω est locale. Pour tout $x \in M$ il existe un ouvert U_x contenant x et un $F(U_x)$-sous-module de rang 1 de Ω sur U_x, tels que la restriction d'une dérivation de Ω sur U_x est la dérivée de Lie par rapport à un champ de vecteurs ψ_x sur U_x. Alors, il existe un et un seul $\psi \in \chi(M)$ tel que sa restriction sur U_x est ψ_x pour tout $x \in M$. Ainsi, on peut montrer que toute dérivation de Ω est la dérivée de Lie par rapport à un et un seul champ de vecteurs appartenant à son normalisateur $\mathfrak{N}$ dans $\chi(M)$. Le centralisateur de Ω est nul, l'idéal dérivé de Ω coïncide à Ω. Par conséquent, Ω est un idéal caractéristique de $\mathfrak{N}$. De plus, on montre que toute dérivation du normalisateur $\mathfrak{N}$ est intérieure. Ainsi, l'application qui à X de $\mathfrak{N}$ fait correspondre la dérivée de Lie par rapport à X, est un isomorphisme d'algèbres de Lie de $\mathfrak{N}$ dans l'algèbre de Lie de toute dérivation de Ω. Alors, le premier espace de cohomologie de Chevalley-Eilenberg de Ω (resp. de $\mathfrak{N}$) est isomorphe à $\mathfrak{N}/\Omega$ (resp. à $\{0\}$). On utilise ces résultats pour calculer le premier espace de cohomologie de Chevalley-Eilenberg de l'algèbre de Lie des champs de vecteurs à support compact et des algèbres de Lie attachées à un feuilletage généralisé, car elles sont des distributions involutives sur M. Si la variété M est munie d'une structure de feuilletage régulier, Ω est l'algèbre de Lie des champs de vecteurs tangents au feuilletage, alors certains résultats correspondants de [4] et de [6] découlent de nos résultats. Dans un autre article, on considérera d'autres applications sur les algèbres de Lie des champs de vecteurs attachées à une connexion et définies à la manière de [5].

2 Etude des dérivations des algèbres de Lie d'une distribution involutive de *M*

Dans toute la suite, M est une variété différentiable de classe C^∞ de dimension n, tous les objets utilisés sont supposés C^∞ sur M. Dans cette section, $F(M)$ est l'anneau des fonctions réelles de M, $\chi(M)$ désigne l'algèbre de Lie des champs de vecteurs sur M avec son crochet habituel. Ω est une distribution involutive de M, autrement dit un $F(M)-$sous-module de $\chi(M)$ stable par le crochet de champs de vecteurs. Dans la suite, sauf mention expresse ; pour tout $x \in M$, il existe un champ $X \in \Omega$ tel que $X(x) \neq 0$. On note par L_X la dérivée de Lie par rapport à $X \in \chi(M)$, et $\mathrm{Supp}(X)$ le support de X sur M. On adopte la convention d'Einstein sur la sommation d'indices, sauf mention expresse.

Définition 2.1. Une $\mathbb{R}$-dérivation D de Ω est une application $\mathbb{R}-$linéaire de Ω dans Ω telle que $\forall X,Y \in \Omega,\ D[X,Y] = [D(X),Y] + [X,D(Y)]$.

Dans cette section, une $\mathbb{R}$-dérivation de Ω est tout simplement appelée dérivation de Ω.

Proposition 2.2. *Soient U un ouvert de M contenant x, et $X \in \chi(M)$ tel que $X(x) \neq 0$, alors il existe $Y \in \Omega$ avec* $\mathrm{Supp}(Y) \subset U$ *tel que* $[X,Y](x) \neq 0$.

Démonstration. Soit $X \in \chi(M)$ vérifiant les hypothèses ci-dessus. Par le théorème classique de Fröbenius appliqué à $\{X\}$ dans un ouvert contenu dans U, on réduit l'équation aux dérivées partielles correspondante à une forme plus facile à résoudre. Ce qui donne l'existence du champ Y de Ω à support contenu dans U en utilisant les astuces des fonctions plateaux. □

Définition 2.3. Une dérivation D de Ω est dite locale si pour tout ouvert non vide U de M et $X \in \Omega$ tels que $X_{|U} \equiv 0$, on a $D(X)_{|U} \equiv 0$.

Proposition 2.4. *Toute dérivation de Ω est locale.*

Démonstration. Par la Proposition 2.2, on peut adapter la démonstration dans [8]. □

Proposition 2.5. *Soient une carte (U,φ) de système de coordonnées $(x^i)_i$, et $\psi, \psi', \psi'', \underset{i\neq n}{a^i \frac{\partial}{\partial x^i}}$ appartenant à $\chi(U)$:*

1. *Soit $x \in U$ tel qu'il existe $i_0 \neq n$, avec $a^{i_0}(x) \neq 0$. Si $\forall f \in F(U)$:*

$$L_\psi\left(f\left(\underset{i\neq n}{a^i\frac{\partial}{\partial x^i}}+\frac{\partial}{\partial x^n}\right)\right)=L_{\psi'}\left(\underset{i\neq n}{fa^i\frac{\partial}{\partial x^i}}\right)+L_{\psi''}\left(f\frac{\partial}{\partial x^n}\right) \tag{2.1}$$

   Alors il existe un voisinage V de x dans U, où $\psi \underset{V}{=} \psi' \underset{V}{=} \psi''$.

2. *On désigne par Pr_n la n-ième projection de $\mathbb{R}^n$. Soient $x_0 \in U$ tel que $a^i(x_0) = 0\ \forall i \neq n$ et $g_{x_0} : x \in U \mapsto g_{x_0}(x) = x^n - \mathrm{Pr}_n(\varphi(x_0))$. Si $\forall f \in F(U)$:*

$$\begin{aligned} L_\psi\left(f\left(\underset{i\neq n}{a^i\frac{\partial}{\partial x^i}}+(g_{x_0}+1)\frac{\partial}{\partial x^n}\right)\right) &= L_{\psi'}\left(f\left(\underset{i\neq n}{a^i\frac{\partial}{\partial x^i}}+\frac{\partial}{\partial x^n}\right)\right)+ \\ &\quad + L_{\psi''}\left(fg_{x_0}\frac{\partial}{\partial x^n}\right) \end{aligned} \tag{2.2}$$

   Alors $\psi(x_0) = \psi'(x_0) = \psi''(x_0)$.

Démonstration. Pour avoir le résultat de *1.*, on utilise la relation (2.1) en remplaçant f par des polynômes convenables sur un voisinage V de x, tel que pour tout $y \in V$, $a^{i_0}(y) \neq 0$. Le résultat de *2.* est obtenu en travaillant sur un voisinage W_{x_0} de x_0, où pour tout $y \in W_{x_0}$, $g_{x_0}(y) \neq -1$; et en remplaçant f de la relation (2.2) par des polynômes et exponentielle convenables. □

Proposition 2.6. *Soient D une dérivation de Ω et U un domaine d'une carte tels qu'il existe un $F(U)$-sous-module Γ_U de rang 1 de Ω_U et $\psi \in \chi(U)$ avec $D_{U|\Gamma_U} = L_\psi$. Si pour tout $x \in U$, il existe un ouvert $V \subset U$ contenant x tel que pour tout $F(V)$-sous-module Δ_V de rang 1 de Ω_V, il existe $\zeta \in \chi(V)$ avec $D_{V|\Delta_V} = L_\zeta$; alors $D_U = L_\psi$ et réciproquement.*

Démonstration. Soient D une dérivation de Ω et U un domaine d'une carte contenant z. D'après la Proposition 2.4, D_U est une dérivation de Ω_U. On peut considérer un système de coordonnées $(x^1,\ldots,x^{n-1},y)$ de U tel que Γ_U est le $F(U)$-module engendré par $\frac{\partial}{\partial y}$. Soit $Y = \underset{i\neq n}{a^i\frac{\partial}{\partial x^i}} + a^n\frac{\partial}{\partial y} \in \Omega_U$, alors $Z = \underset{i\neq n}{a^i\frac{\partial}{\partial x^i}} \in \Omega_U$. Procédons en 3 étapes :

1) Si $Z = \underset{i\neq n}{a^i} \frac{\partial}{\partial x^i}$, où il existe $i_0 \neq n$ tel que $a^{i_0}(z) \neq 0$:

En utilisant le théorème classique de Fröbenius sur les deux champs $Z + \frac{\partial}{\partial y}$ et Z, on trouve un ouvert $V \subset U$ contenant z et deux $F(V)$-sous-modules correspondants de rang 1 de Ω_V. La restriction de D_V sur ces deux modules correspond respectivement à la dérivée de Lie par rapport à ζ et ζ' de $\chi(V)$. En appliquant *1.* de la Proposition 2.5, il existe un voisinage W_z de z tel que $\zeta \underset{W_z}{=} \zeta' \underset{W_z}{=} \psi$.

2) Si $Z = \underset{i\neq n}{a^i} \frac{\partial}{\partial x^i}$ avec $a^i(z) = 0\ \forall i \neq n$:

Soit la fonction g_z dans la Proposition 2.5, d'une façon analogue à 1), on utilise le théorème classique de Fröbenius sur les deux champs $Z + (g_z + 1)\frac{\partial}{\partial y},\ Z + \frac{\partial}{\partial y}$. Il existe un ouvert $V \subset U$ contenant z tel que la restriction de D_V sur les deux modules correspondants est respectivement égale à la dérivée de Lie par rapport à ζ et ζ' de $\chi(V)$. On applique *2.* de la Proposition 2.5, et on trouve $\zeta(z) = \zeta'(z) = \psi(z)$.

3) En utilisant la $\mathbb{R}$-linéarité de D_U et en combinant les deux cas précédents pour tout $z \in M$, on a $D_U(Y) = L_\psi(Y)$ pour tout $Y \in \Omega_U$. Ainsi, on obtient $D_U = L_\psi$.

La réciproque est immédiate. □

Théorème 2.7. *Toute dérivation de l'algèbre de Lie Ω est une dérivée de Lie par rapport à un et un seul champ de vecteurs sur M.*

Démonstration. Soit $x \in M$, il existe $X \in \Omega$ tel que $X(x) \neq 0$. D'après le théorème classique de Fröbenius, il existe une carte (U_x, φ_x) contenant x et un système de coordonnées locales $(x^1, \ldots, x^{n-1}, y)$ correspondant, tels que $X \underset{U_x}{=} \frac{\partial}{\partial y}$.

Soit D une dérivation de Ω qui est locale par la Proposition 2.4, donc D_{U_x} est une dérivation de Ω_{U_x}. Soit $f \in F(U_x)$, comme $\frac{\partial}{\partial y} \in \Omega_{U_x}$, alors on peut écrire d'une manière unique :

$$D_{U_x}\left(f\frac{\partial}{\partial y}\right) = D^0(f)\frac{\partial}{\partial y} + \underset{0<a<n}{D^a(f)}\frac{\partial}{\partial x^a} \tag{2.3}$$

Ω étant une distribution involutive de M, alors $\left[\frac{\partial}{\partial y}, f\frac{\partial}{\partial y}\right] \in \Omega_{U_x}$. Par la Définition 2.1, on obtient :

$$D_{U_x}\left[\frac{\partial}{\partial y}, f\frac{\partial}{\partial y}\right] = \left[D_{U_x}\left(\frac{\partial}{\partial y}\right), f\frac{\partial}{\partial y}\right] + \left[\frac{\partial}{\partial y}, D_{U_x}\left(f\frac{\partial}{\partial y}\right)\right]$$

On a

$$\left[D_{U_x}\left(\frac{\partial}{\partial y}\right), f\frac{\partial}{\partial y}\right] = \left(D^0(1)\frac{\partial f}{\partial y} + D^a(1)\frac{\partial f}{\partial x^a} - f\frac{\partial D^0(1)}{\partial y}\right)\frac{\partial}{\partial y} - f\frac{\partial D^a(1)}{\partial y}\frac{\partial}{\partial x^a}$$

et

$$\left[\frac{\partial}{\partial y}, D_{U_x}\left(f\frac{\partial}{\partial y}\right)\right] = \frac{\partial D^0(f)}{\partial y}\frac{\partial}{\partial y} + \frac{\partial D^a(f)}{\partial y}\frac{\partial}{\partial x^a}$$

Donc

$$\begin{aligned} D_{U_x}\left[\frac{\partial}{\partial y}, f\frac{\partial}{\partial y}\right] &= \left(D^0(1)\frac{\partial f}{\partial y} + D^a(1)\frac{\partial f}{\partial x^a} - f\frac{\partial D^0(1)}{\partial y} + \frac{\partial D^0(f)}{\partial y}\right)\frac{\partial}{\partial y} + \\ &\quad + \left(\frac{\partial D^a(f)}{\partial y} - f\frac{\partial D^a(1)}{\partial y}\right)\frac{\partial}{\partial x^a} \end{aligned}$$

Or

$$\left[\frac{\partial}{\partial y}, f\frac{\partial}{\partial y}\right] = \frac{\partial f}{\partial y}\frac{\partial}{\partial y}$$

Alors

$$D^0(1)\frac{\partial f}{\partial y} + D^a(1)\frac{\partial f}{\partial x^a} - f\frac{\partial D^0(1)}{\partial y} = D^0\left(\frac{\partial f}{\partial y}\right) - \frac{\partial D^0(f)}{\partial y} \tag{2.4}$$

$$\frac{\partial D^a(f)}{\partial y} - f\frac{\partial D^a(1)}{\partial y} = D^a\left(\frac{\partial f}{\partial y}\right) \tag{2.5}$$

, où $a = 1, \ldots, n-1$.
$\forall j = 1, \ldots, n$; soit x_1^j un point arbitraire de $\mathrm{Pr}_j(\varphi_x(U_x))$ avec $x^n = y$, on a :

$$D_{U_x}\left[(x^j - x_1^j)\frac{\partial}{\partial y}, f\frac{\partial}{\partial y}\right] = \left[D_{U_x}\left((x^j - x_1^j)\frac{\partial}{\partial y}\right), f\frac{\partial}{\partial y}\right] + \left[(x^j - x_1^j)\frac{\partial}{\partial y}, D_{U_x}\left(f\frac{\partial}{\partial y}\right)\right]$$

, et d'une manière analogue aux (2.4) et (2.5), on a :

$$\begin{aligned} D^0\left((x^j - x_1^j)\frac{\partial f}{\partial y}\right) &= D^0(x^j - x_1^j)\frac{\partial f}{\partial y} + D^i(x^j - x_1^j)\frac{\partial f}{\partial x^i} - \delta_i^j D^i(f) - f\frac{\partial D^0(x^j - x_1^j)}{\partial y} + \\ &\quad + (x^j - x_1^j)\frac{\partial D^0(f)}{\partial y} \end{aligned} \tag{2.6}$$

$$D^a\left((x^j - x_1^j)\frac{\partial f}{\partial y}\right) = (x^j - x_1^j)\frac{\partial D^a(f)}{\partial y} - f\frac{\partial D^a(x^j - x_1^j)}{\partial y} + \delta_n^j D^a(f) \tag{2.7}$$

pour $a = 1, \ldots, n-1$.
En outre, par la relation (2.3) et la Proposition 2.4, chaque D^a est $\mathbb{R}$-linéaire et locale. Alors chaque D^a est un opérateur différentiel d'après un théorème dans [7]. Ainsi on peut écrire :

$$D^0 = \psi = \sum_{|A|\geq 0} \psi^A \frac{\partial^{|A|}}{\partial x^A};\ \text{pour } a \in \{1, \ldots, n-1\}, D^a = \sum_{|A|\geq 0} \eta_a^A \frac{\partial^{|A|}}{\partial x^A}$$

, où $A = (A_1, \ldots, A_n) \in \mathbb{N}^n$ avec $|A| = \sum_{i=1}^n A_i$, $\frac{\partial^{|A|}}{\partial x^A} = \frac{\partial^{|A|}}{(\partial x^1)^{A_1} \ldots (\partial x^{n-1})^{A_{n-1}} (\partial y)^{A_n}}$ et les ψ^A, η_a^A appartiennent à $F(U_x)$.
En remplaçant f dans les relations (2.4), (2.5), (2.6) et (2.7) par des polynômes de degré quelconque s'annulant en un point arbitraire de U_x, on prouve :

$$D^0(1) = -\frac{\partial \psi^{(0,\ldots,0,1)}}{\partial y} = \psi^{(0,\ldots,0)}, D^a(1) = -\frac{\partial \psi^{(a,0,\ldots,0)}}{\partial y} \text{ pour } a = 1, \ldots, n-1$$

et D^0 (resp. D^a $a = 1, \ldots, n-1$) est un opérateur différentiel d'ordre ≤ 1 (resp. d'ordre 0).
Dans toute la suite, $\psi^{(0,\ldots,0,1)}$ (resp. $\psi^{(a,0,\ldots,0)}$) est noté par $\psi^{0,1}$ (resp. $\psi^{a,0}$).
On écrit

$$\psi = \psi_0 + \psi_{1_x}, \text{ où } \psi_0 = D^0(1) \text{ et } \psi_{1_x} = \psi^{0,1}\frac{\partial}{\partial y} + \psi^{a,0}\frac{\partial}{\partial x^a} \in \chi(U_x)$$

Ainsi pour $f \in F(U_x)$

$$\left[\psi_{1_x}, f\frac{\partial}{\partial y}\right] = \left(\psi^{0,1}\frac{\partial f}{\partial y} + \psi^{a,0}\frac{\partial f}{\partial x^a} - f\frac{\partial \psi^{0,1}}{\partial y}\right)\frac{\partial}{\partial y} - f\frac{\partial \psi^{a,0}}{\partial y}\frac{\partial}{\partial x^a}$$

Alors

$$\begin{aligned}\left[\psi_{1_x}, f\frac{\partial}{\partial y}\right] + \left(f\frac{\partial \psi^{a,0}}{\partial y} + D^a(f)\right)\frac{\partial}{\partial x^a} &= \left(-f\frac{\partial \psi^{0,1}}{\partial y} + \psi^{0,1}\frac{\partial f}{\partial y} + \psi^{a,0}\frac{\partial f}{\partial x^a}\right)\frac{\partial}{\partial y} + \\ &\quad + D^a(f)\frac{\partial}{\partial x^a}\end{aligned}$$

et

$$\left[\psi_{1_x}, f\frac{\partial}{\partial y}\right] + \left(f\frac{\partial \psi^{a,0}}{\partial y} + D^a(f)\right)\frac{\partial}{\partial x^a} = (\psi_0 + \psi_{1_x})(f)\frac{\partial}{\partial y} + D^a(f)\frac{\partial}{\partial x^a}$$

Donc

$$D_{U_x}\left(f\frac{\partial}{\partial y}\right) = \left[\psi_{1_x}, f\frac{\partial}{\partial y}\right] + \left(f\frac{\partial \psi^{a,0}}{\partial y} + D^a(f)\right)\frac{\partial}{\partial x^a}$$

Mais chaque D^a $a = 1, \dots, n-1$ est d'ordre 0 et $D^a(1) = -\frac{\partial \psi^{a,0}}{\partial y}$ $\forall a \neq 0, n$; alors

$$f\frac{\partial \chi^{a,0}}{\partial y} + D^a(f) = 0 \ \forall a \neq 0, n$$

Donc

$$D_{U_x}\left(f\frac{\partial}{\partial y}\right) = \left[\psi_{1_x}, f\frac{\partial}{\partial y}\right] \ \forall f \in F(U_x)$$

Pour tous $z \in U_x$ et $Y \in \Omega$ tels que $Y(z) \neq 0$, on peut répéter le même raisonnement utilisé pour X à Y et on a :
Il existe U_z contenant z, un domaine d'une carte dans U_x de système $(x'^i)_i$ tel que $Y = \frac{\partial}{\partial x'^n}$ et $\exists \zeta \in \chi(U_z)$ tel que $D_{U_z}\left(f\frac{\partial}{\partial x'^n}\right) = L_\zeta\left(f\frac{\partial}{\partial x'^n}\right)$ $\forall f \in F(U_z)$. D'après la Proposition 2.6, on a $D_{U_x} = L_{\psi_{1_x}}$ pour tout $x \in M$. Donc $(U_x, \varphi_x)_{x \in M}$ forme un atlas de M, et pour tout $x \in M$, il existe $\psi_{1_x} \in \chi(U_x)$ avec $D_{U_x} = L_{\psi_{1_x}}$. Pour tous $x, y \in M$ tels que $U_x \cap U_y \neq \varnothing$, $\psi_{1_x} \underset{U_x \cap U_y}{=} \psi_{1_y}$ par la Proposition 2.6. Ainsi

$$\exists \psi \in \chi(M) \text{ tel que } D = L_\psi \text{ avec } \forall x \in M,\ \psi_{|U_x} = \psi_{1_x}$$

Soit $y \in M$, il existe $X \in \Omega$ tel que $X(y) \neq 0$. En utilisant le théorème classique de Fröbenius sur $\{X\}$ et en faisant le crochet de ψ avec fX , pour tout $f \in F(M)$, on obtient l'unicité de ψ en y quelconque de M. □

Proposition 2.8. *Le centralisateur $\mathfrak{C}$ de Ω dans $\chi(M)$ est réduit à zéro.*

Démonstration. Par définition, $\mathfrak{C} = \{X \in \chi(M) \text{ tel que } [X, \Omega] = \{0\}\}$.
Soit $X \in \mathfrak{C} \neq \varnothing$ car $0 \in \mathfrak{C}$, raisonnons par l'absurde.
On suppose que $X \neq 0$, donc il existe $x \in M$ tel que $X(x) \neq 0$. D'après la Proposition 2.2, il existe un ouvert U contenant x et $Y \in \Omega$ tels que $[X, Y](x) \neq 0$. Ce qui contredit $X \in \mathfrak{C}$, donc $X \equiv 0$; d'où $\mathfrak{C} = \{0\}$. □

Proposition 2.9. *L'idéal dérivé de Ω coïncide à Ω.*

Démonstration. L'idéal dérivé de Ω noté par $[\Omega,\Omega]$ est l'algèbre de Lie engendrée par tous les crochets de deux éléments de Ω.
$[\Omega,\Omega] \neq \varnothing$ car $0 \in [\Omega,\Omega]$. Comme Ω est une algèbre de Lie, alors $[\Omega,\Omega] \subset \Omega$.
Pour montrer que $\Omega \subset [\Omega,\Omega]$, raisonnons par l'absurde.
Supposons qu'il existe $X \in \Omega - \{0\}$ tel que $X \notin [\Omega,\Omega]$, alors il existe $x \in M$ tel que $X(x) \neq 0$ et $X(x) \notin [\Omega,\Omega](x)$. Par le théorème classique de Fröbenius dans un ouvert convenable U contenant x, il existe $f \in F(U)$ telle que $Xf \underset{U}{=} 1$. Donc, on a $[X,fX] \underset{U}{=} X$ et $[X,fX](x) = X(x)$. Alors $[X,fX](x) \in [\Omega,\Omega](x)$ car Ω est une distribution involutive de M. Ce qui contredit $X(x) \notin [\Omega,\Omega](x)$, ainsi $\Omega \subset [\Omega,\Omega]$; d'où $[\Omega,\Omega] = \Omega$. □

Remarque 2.10. Cette dernière proposition est toujours vraie pour toute distribution involutive de M, c'est à dire pour tout $F(M)$-sous-module de $\chi(M)$ stable par le crochet habituel.

Définition 2.11. Le normalisateur $\mathfrak{N}$ de Ω est $\{X \in \chi(M)$ tel que $[X,\Omega] \subset \Omega\}$.

Théorème 2.12. *Toute dérivation de $\mathfrak{N}$ est intérieure.*

Démonstration. Ω est un idéal caractéristique du normalisateur $\mathfrak{N}$ si elle est stable par toute dérivation de $\mathfrak{N}$.
Par la Proposition 2.9 et un résultat classique cf. [1], on prouve que Ω est un idéal caractéristique de $\mathfrak{N}$. En utilisant le Théorème 2.7, la Proposition 2.9, la Proposition 2.8 et en adaptant une preuve de [6] pp.68-69 ; on montre que toute dérivation de $\mathfrak{N}$ est intérieure. □

Définition 2.13. Soient $(A,[,]_A)$ et $(B,[,]_B)$ deux $\mathbb{R}-$algèbres de Lie. Un homomorphisme d'algèbres de Lie de A dans B est une application $\mathbb{R}$-linéaire h telle que pour tous X,Y de A, $h([X,Y]_A) = [h(X),h(Y)]_B$. C'est un isomorphisme si h est bijective.

Corollaire 2.14. *On note par* $\mathrm{Der}(A)$ *l'ensemble de toute dérivation d'une algèbre de Lie A. Les algèbres de Lie* $\mathrm{Der}(\Omega)$, $\mathfrak{N}$, $\mathrm{Der}(\mathfrak{N})$ *sont isomorphes et le premier espace de cohomologie de Chevalley-Eilenberg de Ω (resp. de $\mathfrak{N}$) est isomorphe à l'algèbre de Lie $\mathfrak{N}/\Omega$ (resp. à $\{0\}$).*

Démonstration. On note par ad_Ω l'ensemble des dérivations intérieures de Ω. Par définition, $\mathrm{H}^1(\Omega) = \mathrm{Der}(\Omega)/\mathrm{ad}_\Omega$ cf. [10].
Soit l'application $\mathbb{R}$-linéaire $\theta : \mathfrak{N} \underset{X \longmapsto \theta_X = L_X}{\longrightarrow} \mathrm{Der}(\Omega)$, où le crochet dans $\mathrm{Der}(\Omega)$ est défini par $\forall D,D'$; $[D,D'] = D \circ D' - D' \circ D$ avec $\circ$ la loi de composition des applications. L'application θ est bijective en utilisant la Proposition 2.8, le Théorème 2.7 ; et un homomorphisme par l'identité de Jacobi. D'une manière analogue, on a les deux résultats : $\Omega \cong \mathrm{Der}(\Omega)$ et, $\mathfrak{N} \cong \mathrm{Der}(\mathfrak{N})$ en utilisant le Théorème 2.12. Ainsi $\mathrm{Der}(\Omega) \cong \mathfrak{N} \cong \mathrm{Der}(\mathfrak{N})$.
Par isomorphisme, on obtient $\mathrm{H}^1(\Omega) \cong \mathfrak{N}/\Omega$ et $\mathrm{H}^1(\mathfrak{N}) \cong \{0\}$. □

Remarque 2.15. Supposons que l'algèbre de Lie Ω n'est pas identiquement nulle et l'ensemble $\mathfrak{O} = \{x \in M$ tel que $\Omega(x) \neq 0\}$ est différent de M. La Proposition 2.4 reste valable en utilisant un résultat analogue sur la sous-variété ouverte $\mathfrak{O}$. En s'inspirant d'une preuve faite dans [6] pp.458-463, si $\mathfrak{O}$ est partout dense dans M, alors :
D'après le Théorème 2.7, pour tout $D \in \mathrm{Der}(\Omega)$ il existe $X \in \chi(\mathfrak{O})$ tel que $D_{|\mathfrak{O}} = L_X$. Si le

prolongement $\overline{X}$ de chaque X correspondant à $D \in \mathrm{Der}(\Omega)$, appartient à $\mathfrak{N}$; alors $D = L_{\overline{X}}$. Le centralisateur de Ω est nul, ainsi le Corollaire 2.14 reste valable.

Exemple 2.16. Sur $\mathbb{R}^3$, l'ensemble $\mathfrak{B}$ des champs de vecteurs de la forme $f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$ avec $f, g, h \in F\left(\mathbb{R}^3\right)$, s'annulant en $(0,0,0)$, est une distribution non-régulière, involutive. Le normalisateur de $\mathfrak{B}$ est $\mathfrak{B}$ lui-même, le premier espace de cohomologie de Chevalley-Eilenberg $\mathrm{H}^1(\mathfrak{B}) \cong \{0\}$.

3 Application à l'algèbre de Lie des champs de vecteurs à support compact et aux algèbres de Lie attachées à un feuilletage généralisé

Soit $\mathfrak{C}_c$ l'ensemble des champs de vecteurs à support compact sur M, il est facile de vérifier que $\mathfrak{C}_c$ est une distribution involutive de M.

Théorème 3.1. *L'idéal dérivé de l'algèbre de Lie $\mathfrak{C}_c$ coïncide à $\mathfrak{C}_c$, son centralisateur est nul et son premier espace de cohomologie de Chevalley-Eilenberg est isomorphe à $\chi(M)/\mathfrak{C}_c$.*

Démonstration. On peut montrer que pour tous $X \in \chi(M)$ et $Y \in \mathfrak{C}_c$:

$$[X,Y] \in \mathfrak{C}_c \tag{3.1}$$

Ainsi, le normalisateur de $\mathfrak{C}_c$ dans $\chi(M)$ est $\chi(M)$. Par la Proposition 2.9, $[\mathfrak{C}_c, \mathfrak{C}_c] = \mathfrak{C}_c$, c'est un idéal caractéristique de $\chi(M)$.
Par ailleurs, pour tout $x \in M$, il existe $X \in \mathfrak{C}_c$ tel que $X(x) \neq 0$, donc son centralisateur est nul par la Proposition 2.8. D'après le Corollaire 2.14 :

$$\mathrm{H}^1(\mathfrak{C}_c) \cong \chi(M)/\mathfrak{C}_c$$

□

Définition 3.2. Un feuilletage généralisé $\mathfrak{F} = \{\mathfrak{F}^\alpha\}_{\alpha\in I}$ sur M est une partition en sous-variétés connexes de $M = \bigcup\limits_{\alpha\in I} \mathfrak{F}^\alpha$ qui sont exactement les orbites des compositions des flots engendrés par les champs de vecteurs localement tangents aux feuilles de $\mathfrak{F}$ cf. [2].

Dans toute la suite, M est une variété différentiable munie d'une structure de feuilletage généralisé $\mathfrak{F}$.

Définition 3.3. L'algèbre de Lie des champs de vecteurs tangents au feuilletage $\mathfrak{F}$ est désignée par $\chi(\mathfrak{F})$. Le normalisateur de $\chi(\mathfrak{F})$ dans $\chi(M)$ est noté par $\mathfrak{N}(\mathfrak{F})$. On désigne par $\mathfrak{L}(\mathfrak{F})$ l'algèbre de Lie des champs de vecteurs qui préservent le feuilletage $\mathfrak{F}$, $\mathfrak{L}(\mathfrak{F})$ est contenu dans $\mathfrak{N}(\mathfrak{F})$ cf. [2].

Exemple 3.4. Les feuilletages engendrés par les champs hamiltoniens de l'algèbre de Lie locale de Kirillov cf. [3] ou ceux engendrés par la structure de Jacobi ou de Poisson, sont des feuilletages généralisés. Ils définissent un $\chi(\mathfrak{F})$ finiment engendré cf. [2].
La structure hamiltonienne dans [9] définit un feuilletage généralisé aux feuilles présymplectiques dont l'algèbre de Lie $\chi(\mathfrak{F})$ y est détaillée.

Théorème 3.5. *L'algèbre de Lie* $\chi_c(\mathfrak{F}) = \chi(\mathfrak{F}) \cap \mathfrak{C}_c$ *est un idéal caractéristique de* $\chi(\mathfrak{F})$ *tel que* $[\chi_c(\mathfrak{F}), \chi_c(\mathfrak{F})] = \chi_c(\mathfrak{F})$. *Si* $\forall x \in M$, $\exists X \in \chi_c(\mathfrak{F})$ *tel que* $X(x) \neq 0$, *alors le centralisateur de* $\chi_c(\mathfrak{F})$ *est nul et* $\mathrm{H}^1(\chi_c(\mathfrak{F})) \cong \mathfrak{N}(\mathfrak{F}) / \chi_c(\mathfrak{F})$.

Démonstration. Comme $\chi(\mathfrak{F})$, $\mathfrak{C}_c$ sont des distributions involutives sur M ; alors $\chi_c(\mathfrak{F}) = \chi(\mathfrak{F}) \cap \mathfrak{C}_c$ l'est aussi. Par la Proposition 2.9, $[\chi_c(\mathfrak{F}), \chi_c(\mathfrak{F})] = \chi_c(\mathfrak{F})$; c'est un idéal caractéristique de $\chi(\mathfrak{F})$. Par ailleurs $\chi_c(\mathfrak{F}) \subset \chi(\mathfrak{F})$, alors $[\mathfrak{N}(\mathfrak{F}), \chi_c(\mathfrak{F})] \subset \chi(\mathfrak{F})$. D'après la relation (3.1), les champs de $[\mathfrak{N}(\mathfrak{F}), \chi_c(\mathfrak{F})]$ sont à support compact, alors $[\mathfrak{N}(\mathfrak{F}), \chi_c(\mathfrak{F})]$ est contenu dans $\chi(\mathfrak{F}) \cap \mathfrak{C}_c = \chi_c(\mathfrak{F})$. On en tire que $\mathfrak{N}(\mathfrak{F})$ est le normalisateur de $\chi_c(\mathfrak{F})$ dans $\chi(M)$.
Le centralisateur de $\chi_c(\mathfrak{F})$ est nul d'après la Proposition 2.8 ; par le Corollaire 2.14, on a :

$$\mathrm{H}^1(\chi_c(\mathfrak{F})) \cong \mathfrak{N}(\mathfrak{F}) / \chi_c(\mathfrak{F})$$

□

Remarque 3.6. Ce dernier théorème est en partie une généralisation dans le cas non régulier d'un théorème dans [6] p.64.

Théorème 3.7. *L'idéal dérivé de* $\chi(\mathfrak{F})$ *est égal à* $\chi(\mathfrak{F})$. *Si le feuilletage est non singulier, c'est à dire que chaque feuille a une dimension superieure ou égale à un, alors le centralisateur de* $\chi(\mathfrak{F})$ *est nul et*

$$\mathrm{H}^1(\chi(\mathfrak{F})) \cong \mathfrak{N}(\mathfrak{F}) / \chi(\mathfrak{F}),\ \mathrm{H}^1(\mathfrak{N}(\mathfrak{F})) \cong \{0\},\ \mathrm{H}^1(\mathfrak{L}(\mathfrak{F})) \cong \mathfrak{N}(\mathfrak{F}) / \mathfrak{L}(\mathfrak{F}).$$

Démonstration. D'après [2], $\chi(\mathfrak{F})$ est une distribution involutive de M. Par la Proposition 2.9, $[\chi(\mathfrak{F}), \chi(\mathfrak{F})] = \chi(\mathfrak{F})$, c'est un idéal caractéristique de $\mathfrak{N}(\mathfrak{F})$.
Si le feuilletage est non singulier, alors $\forall x \in M$, $\exists X \in \chi(\mathfrak{F})$ tel que $X(x) \neq 0$. En utilisant la Proposition 2.8 et le Corollaire 2.14, on a les trois résultats suivants. Par ailleurs, l'algèbre de Lie $\mathfrak{L}(\mathfrak{F})$ contient $\chi(\mathfrak{F})$ et $[\mathfrak{L}(\mathfrak{F}), \chi(\mathfrak{F})] \subset \chi(\mathfrak{F})$ cf. [2]. Ainsi, on adapte une preuve de [6] pp.68-69 et on trouve que toute dérivation de $\mathfrak{L}(\mathfrak{F})$ est une dérivée de Lie par rapport à un champ de $\mathfrak{N}(\mathfrak{F})$. En faisant un raisonnement analogue à celui du Corollaire 2.14, on trouve le dernier résultat. □

Remarque 3.8. Si le feuilletage $\mathfrak{F}$ est singulier, et $\chi(\mathfrak{F})$ vérifie toutes les hypothèses de la *Remarque* 2.15, alors on retrouve les mêmes résultats du Théorème 3.7.

Remarque 3.9. Si le feuilletage $\mathfrak{F}$ est régulier alors on retrouve un théorème de Kanie cf. [4] p.487 et certains résultats de Lichnérowicz cf. [6] p.55, p.64, p.69, à partir du Théorème 3.7.

Remarque 3.10. Si on a l'égalité $\mathfrak{N}(\mathfrak{F}) = \mathfrak{L}(\mathfrak{F})$, alors

$$\mathrm{H}^1(\chi(\mathfrak{F})) \cong \mathfrak{L}(\mathfrak{F}) / \chi(\mathfrak{F}),\ \mathrm{H}^1(\mathfrak{L}(\mathfrak{F})) \cong \{0\}.$$

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MR2774265 (2012f:17041) 17B66 (17B56 53C12)

Randriambololondrantomalala, P. (MAD-UANTS-MIF);
Ravelonirina, H. S. G. (MAD-UANTS-MIF); **Anona, M.** (MAD-UANTS-MIF)

Sur les algèbres de Lie d'une distribution et d'un feuilletage généralisé. (French. English, French summaries) [On the Lie algebras of a distribution and of a generalized foliation]

Afr. Diaspora J. Math. **10** (2010), *no. 2,* 135–144.

Let Ω be an involutive *distribution* on a real differential manifold M, i.e., a subalgebra and a $C^\infty(M)$-submodule of the Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M. It is assumed that Ω satisfies the following condition: for any x in M there exists X in Ω such that $X(x) \neq 0$. The paper states the following facts: (1) Any derivation of Ω is the Lie derivative relative to a unique vector field belonging to the normalizer $\mathcal{N}$ of Ω in $\mathfrak{X}(M)$. (2) The centralizer of Ω is zero. (3) The derived ideal of Ω is Ω. (4) The map $X \mapsto \mathcal{L}_X$ that assigns to X its Lie derivative is an isomorphism of $\mathcal{N}$ onto $\mathrm{Der}(\Omega)$. As an application, the authors determine the first cohomology space, in the sense of Chevalley-Eilenberg, of the Lie algebra of vector fields with compact support (respectively tangent to a generalized foliation) on M.

Reviewed by *Alberto Medina*

ISSN: 1736-4337

Journal of Generalized Lie Theory and Applications

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Journal of Generalized Lie Theory and Applications

Randriambololondrantomalala, J Generalized Lie Theory Appl 2015, S2
http://dx.doi.org/10.4172/1736-4337.S2-e001

Editorial Open Access

A Meeting of Great Minds, Sophus Lie and John Nash throughout their Works

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Keywords: Lie; Nash; Differential geometry

It is well known that Marius Sophus Lie (1842-1899) and John Forbes Nash (1928-2015) are great mathematicians. Sophus Lie comes from Norway and John Nash from United States of America. Their stories have certain resemblances and remarkable relations. This editorial would emphasize some of them. When they have started their university studies, their respective first interests were not mathematics.

That is to say, Lie has been in Astronomy and Nash in Chemical Engineering. Whereas, when they worked on mathematics, the first had Lobatchevski award in 1897 and the second, Nobel prize 1994 and Abel award 2015 (Niels Abel is the uncle of the wife of Sophus Lie: Anna Birch). In addition, their contributions in geometry are considerable, particularly in differential equations. Lie worked on transformation groups relative to partial differential equations, in other words, on Lie groups and on special non-associative algebras named Lie algebras. Nash discovered an important isometrically embedding theorem for a C^k-Riemannian manifold into an Euclidian space (k=1,3,4…), by studying an undetermined partial differential equations. Now, the methods they used offer us an important tool for continuing researches in differential geometry and in other fields cf. [1-9]. Next, these exceptional persons have continued to put more efforts into their mathematics works, even if they had a serious health problem in the middle of their careers. We hope that several mathematicians continue to make profits from results of Lie and Nash for the mathematics's promotion.

This special issue "Recent Advances of Lie Theory in Differential Geometry, in memory of John Nash" honored both Sophus Lie and John Nash as well as their works. John Nash died recently with his wife in a car crash on May 23, 2015. We are grateful regarding their contributions in differential geometry, generally in mathematics.

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Received October 02, 2015; **Accepted** October 12, 2015; **Published** October 14, 2015

Citation: Randriambololondrantomalala P (2015) A Meeting of Great Minds, Sophus Lie and John Nash throughout their Works. J Generalized Lie Theory Appl S2: e001. doi:10.4172/1736-4337.S2-e001

Journal of Generalized Lie Theory and Applications

Towers, J Generalized Lie Theory Appl 2016, S2
http://dx.doi.org/10.4172/1736-4337.S2-e002

Editorial Open Access

Chief Factors of Lie Algebras

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Abstract

In group theory the chief factors allow a group to be studied by its representation theory on particularly natural irreducible modules. It is to be expected, therefore, that they will play an important role in the study of Lie algebras. In this article we survey a few of their properties.

Keywords: L-Algebras; L-Equivalence; c-factor; m-factor; cc-type

Introduction

Throughout L will denote a finite-dimensional Lie algebra over a field F. We call a subalgebra I a *subideal* of a Lie algebra L if there is a chain of subalgebras

$$I = I_0 < I_1 < \ldots < I_n = L,$$

where I_j is an ideal of I_{j+1} for each $0 \le j \le n-1$.

Put $L^1 = L, L^{k+1} = [L^k, L]$ for $k \ge 1$. These are the terms of the lower central series for L. We say that L has *nilpotency class n* if $L^n \ne 0$ but $L^{n+1} = 0$. Let U be a subalgebra of L. If F has characteristic $p>0$ we call U *nilregular* if the nilradical of U, $N(U)$, has nilpotency class less than $p-1$. If F has characteristic zero we regard every subalgebra of L as being nilregular. We say that U is *characteristic in L* if it is invariant under all derivations of L. Nilregular ideals of L have the property that their nilradicals are characteristic in L. Details of the results in this section can be found in studies of Towers [1].

Theorem 1.1.

(i) *If I is a nilregular ideal of L then* $N(I) \subseteq N(L)$.

(ii) *If I is a nilregular subideal of L and every subideal of L containing I is nilregular, then* $N(I) \subseteq N(L)$.

This result was proved by Schenkman [2] for fields of characteristic zero; in characteristic p it follows from a more recent result of Maksimenko [3]. Similarly, we will call the subalgebra U *solregular* if the underlying field F has characteristic zero, or if it has characteristic p and the (solvable) radical of U, $R(U)$, has derived length less than $log_2 p$. Then we have the following corresponding theorem, which uses a result of Petravchuk [4].

Theorem 1.2.

(i) *If I is a solregular ideal of L then* $R(I) \subseteq R(L)$.

(ii) *If I is a solregular subideal of L and every subideal of L containing I is solregular, then* $R(I) \subseteq R(L)$.

These enable us to determine what the minimal ideals of L look like.

Theorem 1.3. *Let L be a Lie algebra over a field F, and let I be a minimal non-abelian ideal of L. Then either*

(i) I is simple or

(ii) F has characteristic p, $N(I)$ has nilpotency class greater than or equal to $p-1$, and $R(I)$ has derived length greater than or equal to $log_2 p$.

As a result of the above we will call the subalgebra U *regular* if it is either nilregular or solregular; otherwise we say that it is *irregular*. Then we have the following corollary.

Corollary 1.4. *Let L be a Lie algebra over a field F. Then every minimal ideal of L is abelian, simple or irregular.*

Block's Theorem on differentiably simple rings [5] describes the irregular minimal ideals as follows.

Theorem 1.5. *Let L be a Lie algebra over a field of characteristic p>0 and let I be an irregular minimal ideal of L. Then* $I \cong S \otimes \mathcal{O}_n$, *where S is simple and* $\mathcal{O}_n$ *is the truncated polynomial algebra in n indeterminates. Moreover, N(I) has nilpotency class* $p-1$ *and R(I) has derived length* $[log_2 p]$.

Primitive Lie algebras

Next we introduce the concept of a primitive Lie algebra. Details of the results in this section can be found in literature of Towers [6]. A word of warning - this terminology has been used for a different concept elsewhere. If U is a subalgebra of L we define U_L, the *core* (with respect to L) of U to be the largest ideal of L contained in U. We say that U is *core-free* in L if $U_L = 0$. We shall call L *primitive* if it has a core-free maximal subalgebra. The *centraliser* of U in L is $C_L(U) = \{x \in L : [x, U] = 0\}$.

There are three types of primitive Lie algebra: *primitive of type* 1 if it has a unique minimal ideal that is abelian; *primitive of type* 2 if it has a unique minimal ideal that is non-abelian; and *primitive of type* 3 if it has precisely two distinct minimal ideals each of which is non-abelian.

Of course, primitive Lie algebras of types 2 and 3 are semisimple, and those of types 1 and 2 are monolithic. (A Lie algebra L is called *monolithic* if it has a unique minimal ideal W, the *monolith* of L.)

Example 2.1. *Examples of each type are easy to find.*

Clearly every primitive solvable Lie algebra is of type 1. *Every simple Lie algebra is primitive of type 2.*

If S is a simple Lie algebra then $L = S \oplus S$ *is primitive of type* 3 *with*

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Received December 23, 2015; **Accepted** January 05, 2016; **Published** January 07, 2016

Citation: Towers DA (2016) Chief Factors of Lie Algebras. J Generalized Lie Theory Appl S2: e002. doi:10.4172/1736-4337.S2-e002

core-free maximal subalgebra $D=\{s+s : s\in S\}$ *the diagonal subalgebra of L.*

Let M be a maximal subalgebra of L. Then M/M_L is a core-free maximal subalgebra of L/M_L. We say that M is

1. *a maximal subalgebra of type* 1 if L/M_L is primitive of type 1;
2. *a maximal subalgebra of type* 2 if L/M_L is primitive of type 2; and
3. *a maximal subalgebra of type* 3 if L/M_L is primitive of type 3.

We say that an ideal A is *complemented* in L if there is a subalgebra U of L such that $L = A + U$ and $A \cap U = 0$. For primitive solvable Lie algebras we have the following analogue of Galois' Theorem for groups.

Theorem 2.2.

1. *If L is a solvable primitive Lie algebra then all core-free maximal subalgebras are conjugate.*
2. *If A is a self-centralising minimal ideal of a solvable Lie algebra L, then L is primitive, A is complemented in L, and all complements are conjugate.*

The *Frattini ideal* of L, $\varphi(L)$, is the core of intersection of the maximal subalgebras of L. We say that L is *φ-free* if $\varphi(L) = 0$. Then we have the following characterisation of primitive Lie algebras of type 1.

Theorem 2.3. *Let L be a Lie algebra over a field F.*

1. *L is primitive of type* 1 *if and only if L is monolithic, with abelian monolith W, and φ-free.*

If F has characteristic zero, then L is primitive of type 1 *if and only if* $L=W\ltimes(C\oplus S)$ *(semi-direct sum), where W is the abelian monolith of L, C is an abelian subalgebra of L, every element of which acts semisimply on W, and S is a Levi subalgebra of L.*

If L is solvable, then L is primitive if and only if it has a self-centralising minimal ideal A.

For type 2 we have

Theorem 2.4.

L is primitive of type 2 *if and only if* $L\cong U+(S\otimes\mathcal{O}_n)$, *where* $S\otimes\mathcal{O}_n$ *is an ideal of L and S is simple.*

If F has characteristic zero, then L is primitive of type 2 *if and only if L is simple.*

L is primitive of type 2 *if and only if there is a primitive Lie algebra X of type* 3 *such that* $L\cong X/B$ *for a minimal ideal B of L.*

For type 3 we have

Theorem 2.5.

L is primitive of type 3 *if and only if L has two distinct minimal ideals* B_1 *and* B_1 *with a common complement and such that the factor algebras* L/B_1 *are primitive of type* 2 *for* i = 1, 2. *Moreover,* B_1 *and* B_1 *are both isomorphic to* $S\otimes\mathcal{O}_n$, *where S is simple.*

If F has characteristic zero, then L is primitive of type 3 *if and only if* $L=S\oplus S$, *where S is simple.*

Chief Factors

The factor algebra A/B is called a *chief factor* of L if B is an ideal of L and A/B is a minimal ideal of L/B. So chief factors are as described in Corollary 1.4 and Theorem 1.5. We can identify different types of chief factor; details for this section can be found in studies of Towers [6]. A chief factor A/B is called *Frattini* if $A/B\subseteq\phi(L/B)$. This concept was first introduced in literature of Towers [7].

If there is a subalgebra, M such that $L = A + M$ and $B\subseteq A\cap M$, we say that A/B is a *supplemented* chief factor of L and that M is a *supplement* of A/B in L Also, if A/B is a non-Frattini chief factor of L, then A/B is *supplemented* by a maximal subalgebra M of L

If A/B is a chief factor of L supplemented by a subalgebra M of L, and $A\cap M=B$ then we say that A/B is *complemented* chief factor of L, and M is a *complement* of A/B in L. When L is *solvable*, it is easy to see that a chief factor is Frattini if and only if it is not complemented. Then we have the following generalisation of the Jordan-Hölder Theorem.

Theorem 3.1. *Let*

$$0<A_1<\ldots<A_n=L \tag{1}$$

$$0<B_1<\ldots<B_n=L \tag{2}$$

be chief series for the Lie algebra L. Then there is a bijection between the chief factors of these two series such that corresponding factors are isomorphic as L-modules and such that the Frattini chief factors in the two series correspond.

The number of Frattini chief factors or of chief factors which are complemented by a maximal subalgebra of a finite-dimensional Lie algebra L is the same in every chief series for L. However, this is not the case for the number of chief factors which are simply complemented in L; in framework of Towers [8] we determine the possible variation in that number.

Note that if L is a primitive Lie algebra of type 3, its two minimal ideals are not L-isomorphic, so we introduce the following concept. We say that two chief factors of L are *L-connected* if either they are L-isomorphic, or there exists an epimorphic image $\overline{L}$ of L which is primitive of type 3 and whose minimal ideals are L-isomorphic, respectively, to the given factors. (It is clear that, if two chief factors of L are L-connected and are not L-isomorphic, then they are nonabelian and there is a single epimorphic image of L which is primitive of type 3 and which connects them.) Then, as we would hope,

Theorem 3.2. *The relation 'is L-connected to' is an equivalence relation on the set of chief factors.*

Let A/B be a supplemented chief factor of L and put $\mathcal{J} = \{M_L : M$ is a maximal subalgebra of L supplementing a chief factor L-connected to $A/B\}$. Let $R=\cap\{N : N\in\mathcal{J}\}$ and $C=A+C_L(A/B)$. Then we call C/R the *crown* of L associated with A/B. This object gives much information about the supplemented chief factors of L.

Theorem 3.3. *Let C/R be the crown associated with the supplemented chief factor A/B of L. Then C/R = Soc(L/R). Furthermore*

(i) *every minimal ideal of L/R is a supplemented chief factor of L which is L-connected to A/B, and*

(ii) *no supplemented chief factor of L above C or below R is L-connected to A/B.*

In other words, there are r ideals $A_1,\ldots, A_r$ *of L such that*

$$C/R=A_1/R\oplus\ldots\oplus A_r/R$$

where A_i/R *is a supplemented chief factor of L which is L-connected to A/B for* $i = 1,\ldots, r$ *and r is the number of supplemented chief factors* *of L which are L-connected to A/B in each chief series for L. Moreover,* $\phi(L/R)=0$.

Corollary 3.4. *Two supplemented chief factors of L define the same crown if and only if they are L-connected.*

Theorem 3.5. *Let L be a solvable Lie algebra, and let* $C/R=\bar{C}$ *be the crown associated with a supplemented chief factor of L. Then* $\bar{C}$ *is complemented in* $\bar{L}$, *and any two complements are conjugate by an automorphism of the form* 1 + *ad a for some* $a \in \bar{C}$.

Finally, in [9], Barnes determined for a solvable Lie algebra which irreducible *L*-modules *A* have the property that $H^1(L,A)=0$.

Theorem 3.6. *Let L be a solvable Lie algebra and let A be an irreducible L-module. Then* $H^1(L,A)=0$ *if and only if L has no complemented chief factor isomrphic to A.*

Covering and Avoidance

The subalgebra *U avoids* the factor algebra A_i/A_{i-1} if $U\cap A_i=U\cap A_{i-1}$; likewise, *U covers* A_i/A_{i-1} if $U+A_i=U+A_{i-1}$. We say that *U* has the covering and avoidance property of *L* if *U* either covers or avoids every chief factor of *L*. We also say that *U* is a *CAP*-subalgebra of *L*. Then these subalgebras give characterisations of solvable and supersolvable Lie algebras; details can be found in studies of Towers [10].

There are a number of ways in which *CAP*-subalgebras arise. For a subalgebra *B* of *L* we denote by [*B* : *L*] the set of all subalgebras *S* of *L* with $B\subseteq S\subseteq L$, and by $[B:L]_{max}$ the set of maximal subalgebras in [*B* : *L*]; that is, the set of maximal subalgebras of *L* containing *B*. We define the set $\mathcal{I}$ by $i\in\mathcal{I}$ if and only if A_i/A_{i-1} is not a Frattini chief factor of *L*. For each $i\in\mathcal{I}$ put

$$\mathcal{M}_i=\{M\in[A_{i-1},L]_{max}:A_i\not\subseteq M\}.$$

Then *U* is a *prefrattini* subalgebra of *L* if

$$U=\bigcap_{i\in\mathcal{I}}M_i \text{ for some } M_i\in\mathcal{M}_i.$$

It was shown in [8] that, when *L* is solvable, this definition does not depend on the choice of chief series, and that the prefrattini subalgebras of *L* cover the Frattini chief factors and avoid the rest; that is, they are *CAP*-subalgebras of *L*.

Further examples were given by Stitzinger [11], where he proved the following result [11] for definitions of the terminology used).

Theorem 4.1. ([11], Theorem 2) *Let* $\mathcal{F}$ *be a saturated formation of solvable Lie algebras, and let U be an* $\mathcal{F}$ *-normaliser of L. Then U covers every* $\mathcal{F}$*-central chief factor of L and avoids every* $\mathcal{F}$*-eccentric chief factor of L.*

The chief factor A_i/A_{i-1} is called *central* if $[L,A_i]\subseteq A_{i-1}$ and *eccentric* otherwise. A particular case of the above result is the following theorem, due to Hallahan and Overbeck.

Theorem 4.2. ([12], Theorem 1) *Let L be a metanilpotent Lie algebra. Then C is a Cartan subalgebra of L if and only if it covers the central chief factors and avoids the eccentric ones.*

A subalgebra *U* of *L* will be called *ideally embedded* in *L* if $I_L(U)$ contains a Cartan subalgebra of *L*, where $I_L(U)=\{x\in L:[x,U]\subseteq U\}$ is the *idealiser* of *U* in *L* . Clearly, any subalgebra containing a Cartan subalgebra of *L* and any ideal of *L* is ideally embedded in *L*. Then we have the following extension of Theorem 4.2.

Theorem 4.3. *Let L be a metanilpotent Lie algebra and let U be ideally embedded in L. Then U is a CAP-subalgebra of L.*

Corollary 4.4. *Let L be any solvable Lie algebra and let U be an ideally embedded subalgebra of L with* $K=N_2(L)\subseteq U$ *. Then U is a CAP-subalgebra of L.*

Another set of examples of *CAP*-subalgebras, which don't require *L* to be solvable, is given by the next result.

Theorem 4.5. *Let L be any Lie algebra, let U be a supplement to an ideal B in L, and suppose that* $B^k\subseteq U$ *for some* $k\in\mathbb{N}$*. Then U is a CAP-subalgebra of L.*

We can calculate the dimension of *CAP*-subalgebras in terms of the chief factors that they cover.

Lemma 4.6. *Let U be a CAP-subalgebra of L, let* $0=A_0<A_1<\ldots<A_n=L$ *be a chief series for L and let* $\mathcal{I}=\{i:1\le i\le n, U \text{ covers } A_i/A_{i-1}\}$ *. Then* $\dim U=\sum_{i\in\mathcal{I}}(\dim A_i-\dim A_{i-1})$.

We have the following characterisations of solvable and supersolvable Lie algebras.

Theorem 4.7. *Every one-dimensional subalgebra of L is a CAP-subalgebra of L if and only if L is supersolvable.*

Theorem 4.8. *Let L be a Lie algebra over any field F. Then L is solvable if and only if all of its maximal subalgebras are CAP-subalgebras.*

Theorem 4.9. *Let L be a Lie algebra over a field F which has characteristic zero, or is algebraically closed field and of characteristic greater than 5. Then L is solvable if and only if there is a maximal subalgebra M of L such that M is a solvable CAP-subalgebra of L.*

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Citation: Towers DA (2016) Chief Factors of Lie Algebras. J Generalized Lie Theory Appl S2: e002. doi:10.4172/1736-4337.S2-e002

Journal of Generalized Lie Theory and Applications

Maakestad, J Generalized Lie Theory Appl 2015, S2
http://dx.doi.org/10.4172/1736-4337.S2-001

Research Article Open Access

Jet Bundles on Projective Space II

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Abstract

In previous papers the structure of the jet bundle as P-module has been studied using different techniques. In this paper we use techniques from algebraic groups, sheaf theory, generliazed Verma modules, canonical filtrations of irreducible SL(V)-modules and annihilator ideals of highest weight vectors to study the canonical filtration $U_l(\mathfrak{g})L^d$ of the irreducible SL(V)-module $H^0(X, \mathcal{O}_X(d))^*$ where $X = \mathbb{G}(m, m+n)$. We study $U_l(\mathfrak{g})L^d$ using results from previous papers on the subject and recover a well known classification of the structure of the jet bundle $\mathcal{P}^l(\mathcal{O}(d))$ on projective space $\mathbb{P}(V^*)$ as P-module. As a consequence we prove formulas on the splitting type of the jet bundle on projective space as abstract locally free sheaf. We also classify the P-module of the first order jet bundle $\mathcal{P}^1_X(\mathcal{O}_X(d))$ for any $d \geq 1$. We study the incidence complex for the line bundle $\mathcal{O}(d)$ on the projective line and show it is a resolution of the ideal sheaf of $I^l(\mathcal{O}(d))$ - the incidence scheme of $\mathcal{O}(d)$. The aim of the study is to apply it to the study of syzygies of discriminants of linear systems on projective space and grassmannians.

Keywords: Algebraic group; Jet bundle; Grassmannian; P-module; Generalized verma module; Higher direct image; Annihilator ideal; Canonical filtration; Discriminant; Koszul complex; Regular sequence; Resolution

Introduction

In a series of papers of Maakestad [1-4], the structure of the jet bundle as P-module has been studied using different techniques. In this paper we continue this study using techniques from algebraic groups, sheaf theory, generalized Verma modules, canonical filtrations of irreducible SL(V)-modules and annihilator ideals of highest weight vectors and study the canonical filtration $U_l(\mathfrak{g})L^d$ of the SL(V)-module $H^0(X, \mathcal{O}_X(d))^*$ where $X = \mathbb{G}(m, m+n)$ is the grassmannian of m-planes in an $m+n$-dimensional vector space. Using results obtained in studies of Maakestad [1] we classify $U_l(\mathfrak{g})L^d$ and as a corollary we recover a well known result on the structure of the jet bundle $\mathcal{P}^l(\mathcal{O}(d))$ on $\mathcal{P}(V^*)$ as P-module. As a consequence we get well known formulas on the splitting type of the jet bundle on projective space as abstract locally free sheaf. We also classify the P-module of the first order jet bundle $\mathcal{P}^1_X(\mathcal{O}_X(d))$ on any grassmannian $X = \mathbb{G}(m, m+n)$ (Corollary 3.10).

In the first section of the paper we study the jet bundle $\mathcal{P}^l_{G/H}(\mathcal{E})$ of any locally free G-linearized sheaf ε on any quotient G/H. Here G is an affine algebraic group of finite type over an algebraically closed field K of characteristic zero and $H \subseteq G$ is a closed subgroup. There is an equivalence of categories between the category of finite dimensional H-modules and the category of finite rank locally free $\mathcal{O}_{G/H}$-modules with a G-linearization. The main result of this section is Theorem 2.3 where we give a classification of the H_l-modules structure of the fiber $\mathcal{P}^l_{G/H}(\mathcal{E})(x)^*$ where $H_l \subseteq H$ is a Levi subgroup. Here $x\,G/H$ is the distinguished K-rational point defined by the identity $e \in G$. We also study the structure of $\mathcal{P}^l_X(\mathcal{O}_X(d))(x)^*$ as H_l-module where $X = \mathbb{G}(m, m+n)$ is the grassmannian of m-planes in an $m+n$-dimensional vector space (Corollary 2.5 and 2.8).

In the second section we study the canonical filtration $U_l(\mathfrak{g})L^d$ for the irreducible SL(V)-module $H^0(\mathbb{G}, \mathcal{O}\mathbb{G}(d))^*$. Here $\mathbb{G} = \mathbb{G}(m, m+n)$. We prove in Theorem 3.5 there is an isomorphism

$$U_l(\mathfrak{g})L^d \cong L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules when $\mathbb{G} = \mathbb{G}(1, n+1) = \mathcal{P}^n$ is projective n-space. As a result we recover in Corollary 3.6 the structure of the fiber $\mathcal{P}^l_{\mathbb{G}}(\mathcal{O}_{\mathbb{G}}(d))(x)^*$ as P-module. This result was proved in another paper [5] using different techniques. We also recover in Corollary 3.8 a known formula on the structure of the jet bundle on projective space as abstract locally free sheaf [2,6-10].

In the third section we study the incidence complex

$$\wedge^{\bullet}\mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))^*_Y$$

of the line bundle $\mathcal{O}(d)$ on the projective line. Using Koszul complexes and general properties of jet bundles we prove it is a locally free resolution of the ideal sheaf of $I^l(\mathcal{O}(d))$ - the incidence scheme of $\mathcal{O}(d)$.

In Appendix A and B we study SL(V)-modules, automorphisms of SL(V)-modules and give an elementary proof of the Cauchy formula.

Hence the paper initiates a general study of the canonical filtration $U_l(g)L^d$ for any line bundle $\mathcal{O}(d)$ with $d \geq 1$ on any grassmannian $\mathbb{G}(m, m+n)$ as P-module. In Section 3 we show some of the complications arising in this study by giving explicit examples.

The study of the jet bundle $\mathcal{P}^l_X(\mathcal{O}_X(d))$ of a line bundle $\mathcal{O}_X(d)$ on the grassmannian $X = \mathbb{G}(m, m+n)$ is motivated partly by its relationship with the discriminant $D^l\mathcal{O}_X(d)$ of the line bundle $\mathcal{O}_X(d)$. There is by studies of Maakestad [11] for all $1 \leq l < d$ an exact sequence of locally free $\mathcal{O}_X$-modules

$$0 \to \mathcal{Q} \to H^0(X, \mathcal{O}_X(d)) \otimes \mathcal{O}_X \to \mathcal{P}^l_X(\mathcal{O}_x(d)) \to 0$$

giving rise to a diagram of maps of schemes

$$\begin{tikzcd} \mathbb{P}(\mathcal{Q}^*) \arrow[r, "i"] \arrow[d, "\pi"] & \mathbb{P}(W^*) \times X \arrow[d, "p"] \\ D^1(\mathcal{O}_X(d)) \arrow[r, "j"] & \mathbb{P}(W^*) \end{tikzcd},$$

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Received October 21, 2015; **Accepted** November 24, 2015; **Published** December 01, 2015

Citation: Maakestad H (2015) Jet Bundles on Projective Space II. J Generalized Lie Theory Appl S2: 001. doi:10.4172/1736-4337.S2-001

Where $W = \mathrm{H}^0(X, \mathcal{O}_X(d))$, π is the restriction of the projection map and i, j are closed immersions. By definition $D^l(\mathcal{O}_X(d)) := \pi(\mathbb{P}(\mathcal{Q}^*))$ is the schematic image of $\mathcal{P}(\mathcal{Q}^*)$ via π. The K-rational points of $\mathcal{P}(\mathcal{Q}^*)$ are pairs of K-rational points (s, x) with the property that $T^l(x)(s) = 0$ in $\mathcal{P}^l_X(\mathcal{O}_X(d))(x)$. The scheme $\mathcal{P}(\mathcal{Q}^*)$ is the *incidence scheme* of the l'th Taylor morphism

$$T^l : \mathrm{H}^0(X, \mathcal{O}_X(d)) \otimes \mathcal{O}_X \to \mathcal{P}^l_X(\mathcal{O}_x(d)).$$

The map π is a surjective generically finite morphism between irreducible schemes. There is by literature of Maakestad [11] a Koszul complex of locally free sheaves on $Y = \mathcal{P}(W^*) \times X$

$$0 \to \mathcal{O}(-r)_Y \otimes \wedge^r \mathcal{P}^l_X(\mathcal{O}_X(d))^*_Y \to \cdots \to \mathcal{O}(-1)_Y \otimes \mathcal{P}^l_X(\mathcal{O}_X(d))^*_Y \to \quad (1.0.1)$$
$$\mathcal{O}_Y \to \mathcal{O}_{\mathbb{P}(\mathcal{Q}^*)} \to 0$$

which is a resolution of the ideal sheaf of $\mathcal{P}(\mathcal{Q}^*)$ when it is locally generated by a regular sequence. The complex 1 might give information on a resolution of the ideal sheaf of $D^l(\mathcal{O}_X(d))$. A resolution of the ideal sheaf of $D^l(\mathcal{O}_X(d))$ will give information on its syzygies. By literature of Maakestad [11] the first discriminant $D^1(\mathcal{O}\mathcal{P}(d))$ on the projective line $\mathcal{P} = \mathcal{P}^1$ is the *classical discriminant of degree d polynomials*, hence it is a determinantal scheme. By the results of Lascoux [12], we get an approach to the study of the syzygies of $D^1(\mathcal{O}\mathcal{P}(d))$. Hence we get two approaches to the study of syzygies of discriminants of line bundles on projective space and grassmannians: One using Taylor maps, incidence schemes, jet bundles and generalized Verma modules. Another one using determinantal schemes.

Jet Bundles on Quotients

In this section we study the jet bundle of any finite rank G-linearized locally free sheaf $\mathcal{E}$ on the grassmannian $G/P = \mathbb{G}(m, m+n)$ as P_l-module, where $P_l \subseteq P$ is a maximal linearly reductive subgroup.

Let K be an algebraically closed field of characteristic zero and let V be a K-vector space of dimension n. Let $H \subseteq G \subseteq \mathrm{GL}(V)$ be closed subgroups. The following holds: There is a quotient morphism

$$\pi : G \to G / H \quad (2.0.2)$$

and G / H is a smooth quasi projective scheme of finite type over K. Moreover

$$H \subseteq G \text{ is parabolic if and only if } G / H \text{ is projective.} \quad (2.0.3)$$

For a proof refer to literature of Jantzen [13]. Let $X = G / H$ and let $\underline{mod}^G(\mathcal{O}_{G/H})$ be the category of locally free $\mathcal{O}_{G/H}$-modules with a G-linearization. Let $\underline{mod}(H)$ be the category of finite dimensional H-modules. It follows from Jantzen [13], there is an exact equivalence of categories

$$\underline{mod}(H) \cong \underline{mod}^G(\mathcal{O}_{G/H}).$$

Let $\mathcal{E} \in \underline{mod}^G(\mathcal{O}_{G/H})$ be a locally free $\mathcal{O}_{G/H}$-module.

Let $Y = G / H \times G / H$ and $p, q : Y \to G / H$ be the canonical projection maps. The scheme G / H is smooth and separated over $\mathrm{Spec}(K)$ hence the diagonal morphism

$$\Delta : G / H \to Y$$

is a closed immersion of schemes. Let $\mathcal{I} \subseteq \mathcal{O}_Y$ be the ideal of the diagonal and let $\mathcal{O}_{\Delta^l} = \mathcal{O}_Y / \mathcal{I}^{l+1}$ be the structure sheaf of the *n'th infinitesimal neigborhood of the diagonal.*

Definition 2.1. *Let $\mathcal{E}$ be a locally free finite rank $\mathcal{O}_{G/H}$-module. Let*

$$\mathcal{P}^l_{G/H}(\mathcal{E}) = p_*(\mathcal{O}_{\Delta^l} \otimes q^*\mathcal{E})$$

be the l'th jet bundle of $\mathcal{E}$.

Proposition 2.2. *There is for all $l \geq 1$ an exact sequence of locally free $\mathcal{O}_{G/H}$-modules*

$$0 \to Sym^l(\Omega^1_{G/H}) \otimes \mathcal{E} \to \mathcal{P}^l_{G/H}(\mathcal{E}) \to^{\phi} \mathcal{P}^{l-1}_{G/H}(\mathcal{E}) \to 0 \quad (2.2.1)$$

with G-linearization.

Proof. By literature of Maakestad [4] sequence 2.2.1 is an exact sequence of locally free $\mathcal{O}_{G/H}$-modules. The scheme Y is equipped with the diagonal G-action. It follows p_* and q^* preserve G-linearizations. We get a diagram of exact sequences of $\mathcal{O}_Y$-modules with a G-linearization

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}^{l+1} \otimes q^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{O}_Y \otimes q^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{O}_{\Delta^l} \otimes q^*\mathcal{E} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{I}^l \otimes q^*\mathcal{E} \arrow[r] & \mathcal{O}_Y \otimes q^*\mathcal{E} \arrow[r] & \mathcal{O}_{\Delta^{l-1}} \otimes q^*\mathcal{E} \arrow[r] & 0 \end{tikzcd}$$

Since p_* preserves G-linearization we get a morphism

$$\phi : \mathcal{P}^l_{G/H}(\mathcal{E}) \to \mathcal{P}^{l-1}_{G/H}(\mathcal{E})$$

preserving the G-linearization, and the Proposition is proved.

Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{h} = Lie(H)$. Let $H_l \subseteq H$ be a Levi subgroup of H. It follows H_l is a maximal linearly reductive subgroup of H. The group H_l is not unique but all such groups are conjugate under automorphisms of H. Let $x \in G / H$ be the K-rational point defined by the identity $e \in G$.

Theorem 2.3. *There is for all $l \geq 1$ an isomorphism*

$$\mathcal{P}^l_X(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus^l_{i=} \mathrm{Sym}^i(\mathfrak{g} / \mathfrak{h})) \quad (2.3.1)$$

of L-modules.

Proof. Dualize the sequence 2.2.1 and take the fiber at x to get the exact sequence

$$0 \to \mathcal{P}^{l-1}_X(\mathcal{E})(x)^* \to \mathcal{P}^l_X(\mathcal{E})(x)^* \to \mathcal{E}(x)^* \otimes \mathrm{Sym}^l(\mathfrak{g} / \mathfrak{h}) \to 0$$

of H-modules (and H_l-modules). This sequence splits since H_l is linearly reductive and the Theorem follows by induction on l.

Hence the study $\mathcal{P}^l_X(\mathcal{E})(x)^*$ as H_l-module is reduced to the study of $\mathcal{E}(x)^*$ and $\mathrm{Sym}^l(\mathfrak{g} / \mathfrak{h})$.

Let $W \subseteq V$ be K-vector spaces of dimension m and $m + n$ and let $G = \mathrm{SL}(V)$ and $P \subseteq G$ the subgroup fixing W. It follows $G / P = \mathbb{G}(m, m + n)$ is the grassmannian of m-planes in V. Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{p} = Lie(P)$. Fix a basis $e_1, .., e_m$ for W and $e_1, .., e_m, e_{m+1}, .., e_{m+n}$ for V. It follows the K-rational points of P are matrices M on the form

$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where $det(A)det(B) = 1$, A an $m \times m$-matrix and B an $n \times n$-matrix. Let $P_l \subseteq P$ be the subgroup defined as follows: The K-rational points of P_l are matrices M on the form

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $det(A)det(B) = 1$ and similarly A an $m \times m$-matrix and B an $n \times n$-matrix. It follows P_l is a Levi subgroup of P, hence it is a maximal linearly reductive subgroup.

Proposition 2.4. *There is a canonical isomorphism*

$$\mathfrak{g} / \mathfrak{p} \cong \mathrm{Hom}(W, V / W)$$

of P-modules.

Proof. By definition $g = sl(V)$, hence $\varphi \in g$ is a map

$\varphi : V \to V$

with $tr(\varphi) = 0$. Let $i : W \to V$ be the inclusion map and $p : V \to V / W$ the projection map. Define the following map:

$J' : \mathfrak{g} \to \mathrm{Hom}(W, V / W)$

by

$j'(\varphi) = p \circ \varphi \ i.$

It follows $j(\mathrm{p}) = 0$ hence we get a well defined map

$j : g/p \to \mathrm{Hom}(W, V / W)$

defined by

$j(\bar{\phi}) = p \circ \phi \circ i.$

One checks g/p and Hom(W, V/W) are P-modules and j a morphism of P-modules. It is an isomorphism and the Proposition follows.

Corollary 2.5. *On $X = \mathbb{G}(m, m + n)$ there is an isomorphism*

$\mathcal{P}^l_X(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus^l_{i=0} \mathrm{Sym}^i(\mathrm{Hom}(W, V / W))$

of P_l -modules.

Proof. The proof follows from Theorem 2.3 and Proposition 2.4.

There is an isomorphism of P-modules

$\mathrm{Hom}(W, V / W) \cong W^* \otimes V / W$

hence the decomposition into irreducible components of the module $\mathrm{Sym}^i(W^* \otimes V / W)$ as P_l -module may be done using the *Cauchy formula* (Appendix B).

Let $\lambda - |i|$ denote λ is a partition of the integer i If $\lambda = \{\lambda_1, .., \lambda_d\}$ is a partition of an integer l, let $\mu(\lambda)$ denote the following partition:

$\mu(\lambda)_i = 1 - \lambda_{d+1-i}.$

Let for any partition λ of an integer l and any vector space W, $\mathbb{S}_\lambda(W)$ denote the *Schur-Weyl module* of λ.

Corollary 2.6. *There is an isomorphism*

$\mathcal{P}^l_X(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus^l_{i=0} (\bigoplus_{\lambda-|i|} \mathbb{S}_\lambda(W^*) \otimes \mathbb{S}_{\mu(\lambda)}(V / W)))$

of SL(W) × SL(V/W)*-modules.*

Proof. By Corollary 2.5 there is an isomorphism

$\mathcal{P}^l_X(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus^l_{i=0} \mathrm{Sym}^i(\mathrm{Hom}(W, V / W))$

of P_l -modules and SL(W) × SL(V/W)-modules, since SL(W) × SL(V/W) ⊆ P_l is a closed subgroup. Since

$\mathrm{Sym}^i(\mathrm{Hom}(W, V / W)) \cong \mathrm{Sym}^i (W^* \otimes V / W)$

the result follows from the Cauchy formula (Appendix B or [14]).

Example 2.7. *Calculation of the cohomology group* $\mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*)$.

In the following we use the notation introduced in litertature of Jantzen [13]. Let $P_{semi} = \mathrm{SL}(m) \times \mathrm{SL}(n) \subseteq P$ be the semi simplification of P. We get a vector bundle

$\pi : G / P_{semi} \to G / P = \mathbb{G}(m, m + n).$

Let $X = G / P$ and $Y = G / P_{semi}$ Given any finite dimensional P-module W, let $\mathcal{L}_X(W)$ denote its corresponding $\mathcal{O}_X$-module. Let W_{semi} denote the restriction of W to P_{semi}. By the results of Perkinson [13] it follows there is an isomorphism

$\pi^* \mathcal{L}_X(W) \cong \mathcal{L}_Y(W_{semi})$

of locally free sheaves. This will help calculating the higher cohomology group

$\mathrm{H}^i(X, \mathcal{L}_X(W))$

since P_{semi} is semi simple and π is a locally trivial fibration. If W is the P-module corresponding to the dual of the j'th exterior power of the jet bundle $\wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*$ we can use this construction to calculate the cohomology group

$\mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*).$

Such a calculation will be by the results of Maakestad [11], Example 5.12 give information on resolutions of the ideal sheaf of $D^l(\mathcal{O}_X(d))$ since the push down of the Koszul complex 1.0.1 is the locally trivial sheaf

$\mathcal{O}(-j) \otimes \mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*).$

To describe the locally trivial sheaf $\mathcal{O}(-j) \otimes \mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*)$ for all i, j we need to calculate the dimension $h^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*)$ and this calculation may be done using the approach indicated above.

Let $m = 2$, $n = 4$ and $X = \mathbb{G}(2, 4)$.

Corollary 2.8. *There is an isomorphism*

$\mathcal{P}^l_X(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus^l_{i=0} \oplus^n_{j=0} \mathrm{Sym}^{2j+m}(W^*) \otimes \mathrm{Sym}^{2j+m}(V / W))$

of SL(2) × SL(2)*-modules. Here* $(n, m) = (\frac{i}{2}, 0)$ *if* $i = 2n$ *and* $(\frac{i-1}{2}, 1)$ *if* $i = 2n + 1$.

Proof. This follows from Corollary 2.5 and Proposition 5.1.

On Canonical Filtrations and Jet Bundles on Projective Space

In this section we study the canonical filtration for the dual of the SL(V)-module of global sections of an invertible sheaf on the grassmannian. We classify the canonical filtration on projective space and as a result recover known formulas on the splitting type of the jet bundle as abstract locally free sheaf.

Let $W \subseteq V$ be vector spaces over K of dimension m and $m + n$. Let W have basis $e_1, .., e_m$ and V have basis $e_1, .., e_{m+n}$. Let V^* have basis $x_1, .., x_{m+n}$. Let $G = \mathrm{SL}(V)$ and $P \subseteq G$ the parabolic subgroup of elements fixing W. It follows there is a quotient morphism

$\pi : G \to G / P$

and $G / P \cong \mathbb{G}(m, m + n)$ is the grassmannian of m-planes in V. Let $\mathbb{P} = \mathbb{G}(1, n+1) = \mathbb{P}(V^*)$. Let $L^d = \mathrm{Sym}^d(\wedge^m W)$. There is an inclusion of P-modules $L^d \subseteq \mathrm{Sym}^d(\wedge^m V)$. Since K has characteristic zero there is an inclusion of G-modules

$\mathrm{H}^0(G / P, \mathcal{O}_{G/P}(d))^* \subseteq \mathrm{Sym}^d(\wedge^m V^*)^* \cong \mathrm{Sym}^d(\wedge^m V).$

Let $g = Lie(G)$ and $p = Lie(P)$. Let U(g) be the universal enveloping algebra og g and let $\mathrm{U}_l(g)$ be the l'th term to its canonical filtration.

By the Corollary 3.11 in studies of Maakestad [15] there is for all $1 \leq l \leq d$ an exact sequence of P-modules

$0 \to \mathcal{P}^l_{\mathbb{G}}(\mathcal{O}_{\mathbb{G}}(d))(x)^* \to \mathrm{H}^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(d))^* \to \mathrm{H}^0(\mathbb{G}, \mathrm{m}^{l+1}\mathcal{O}_{\mathbb{G}}(d))^* \to 0.$

Since the grassmannian is projectively normal in the Plucker embedding we get an inclusion

$$\mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}}(d))^* \subseteq \mathrm{Sym}^d(\wedge^m V)$$

of P-modules. The highest weight vector for $\mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}}(d))^*$ is the line $L^d = \mathrm{Sym}^d(\wedge^m W)$. Let $ann(L^d) \subseteq \mathrm{U}(\mathfrak{g})$ be the left annihilator ideal of L^d. It is the ideal generated by elements $x \in \mathrm{U}(\mathfrak{g})$ with the property $x(L^d) = 0$. Let $ann_l(L^d)$ be its canonical filtration. We get an exact sequence of G-modules

$$0 \to ann(L^d) \otimes L^d \to \mathrm{U}(\mathfrak{g}) \otimes L^d \to \mathrm{H}^0(X,\mathcal{O}_X(d))^* \to 0$$

and an exact sequence of P-modules

$$0 \to ann_l(L^d) \otimes L^d \to \mathrm{U}_l(\mathfrak{g}) \otimes L^d \to \mathrm{U}_l(\mathfrak{g})L^d \to 0$$

for all $l \geq 1$. The G-module $\mathrm{U}(\mathfrak{g}) \otimes L^d$ is the *generalized Verma module* corresponding to the P-module defined by $L^d = \mathrm{Sym}^d(\wedge^m V)$. There is an inclusion of P-modules

$$\mathrm{U}_l(\mathfrak{g})L^d \subseteq \mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}}(d))^*.$$

Definition 3.1. *Let* $\{\mathrm{U}_l(\mathfrak{g})L^d\}_{l\geq 1}$ *be the canonical filtration for* $\mathrm{H}^0(\mathbb{G},\mathcal{O}\mathbb{G}(d))^*$.

Lemma 3.2. *Assume* $y \in \mathfrak{g}$ *and* $x_1 \cdots x_i \in \mathrm{U}_i(\mathfrak{g})$ *with* $x_i \in \mathfrak{g}$. *The following holds:*

$$y(x_1 \cdots x_i) = (x_1 \cdots x_i)y + w$$

where $w \in \mathrm{U}_{i-1}(\mathfrak{g})$ $\omega \in U_{i-1}(\mathfrak{g})$.

Proof. The proof is by induction.

The Lie algebra $\mathfrak{p}$ is the sub Lie algebra of $\mathfrak{g} = \mathrm{sl}(V)$ given by matrices M of the following type:

$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where A is an $m \times m$-matrix, B and $n \times n$-matrix and $tr(A) + tr(B) = 0$. Let $\mathfrak{p}_L$ be the sub Lie algebra of $\mathfrak{p}$ consisting of matrices $M \in \mathfrak{p}$ of the following type:

$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where $tr(A) + tr(B) = 0$.

Proposition 3.3.

The sub Lie algebra $\mathfrak{p}_L \subseteq \mathfrak{p}$ *is a sub P-module of* $\mathfrak{p}$. (3.3.1)

There is an exact sequence of P-modules

$$0 \to \mathfrak{p}/\mathfrak{p}_L \to \mathfrak{g}/\mathfrak{p}_L \to \mathfrak{g}/\mathfrak{p} \to 0. \tag{3.3.2}$$

and $\mathfrak{p}/\mathfrak{p}_L$ *is the trivial P-module.*

The following holds:

$$dim_K(L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L)) = \binom{mn+k}{mn}. \tag{3.3.3}$$

There is a filtration of P-modules

$$0 = G_{l+1} \subseteq G_l \subseteq \cdots \subseteq G_0 = L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L) \tag{3.3.4}$$

with quotients

$$G_i / G_{i+1} \cong L^{d-(l-i)} \otimes \mathrm{Sym}^{l-i}((\mathfrak{g}/\mathfrak{p} \otimes L)$$

for $1 \leq i \leq k$.

Assume $dim_k(W) = 1$ *and let* $W = L$. *There is an exact sequence of P-modules*

$$0 \to \mathfrak{p}_L \otimes L \to \mathfrak{g} \otimes L \to V \to 0 \tag{3.3.5}$$

giving an isomorphism of P-modules $\mathfrak{g}/\mathfrak{p}_L \otimes L \cong V$.

Proof. We prove 3.3.1: In the following A, a are square matrices of size m and b, B square matrices of size n. The K -rational points of the group P are matrices g on the form

$$g = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where $det(A)det(B) = 1$. Assume $x \in \mathrm{p}$ is the following element:

$$x = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$$

with $tr(a) + tr(b) = 0$. It follows $g(x) = gxg^{-1}$ has $tr(gxg^{-1}) = tr(gg^{-1}x) = tr(x) = 0$ hence $gxg^{-1} \in \mathfrak{p}$ and $\mathfrak{p}$ is a P-module. Assume $x \in \mathfrak{p}_L$ ie $tr(a) = tr(b) = 0$. It follows

$$gxg^{-1} = \begin{pmatrix} aAa^{-1} & * \\ 0 & bBb^{-1} \end{pmatrix}$$

and $tr(aAa^{-1}) + tr(aa^{-1}A) = tr(A) = 0$ hence $g(x) \in \mathfrak{p}_L$ and 3.3.1 is proved.

We prove 3.3.2: By 3.3.1 it follows $\mathfrak{p}_L \subseteq \mathfrak{p}$ is a sub P-module. One checks $\mathfrak{p}/\mathfrak{p}_L$ is a trivial P-module. We clearly get an exact sequence of P-modules and 3.3.2 is proved.

We prove 3.3.3: Since

$$dim_K(\mathfrak{g}) = (m + n)^2 - 1 = n^2 + 2mn + m^2 - 1$$

and

$$dim_K(\mathfrak{p}_L) = m^2 + mn + n^2 - 2$$

it follows $dim_k(\mathfrak{g}/\mathfrak{p}_L) = mn + 1$. It follows

$$dim_K(L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes l) = \binom{mn+1+l-1}{mn+1-1} = \binom{mn+l}{mn}.$$

We prove 3.3.4: Since $\mathfrak{p}/\mathfrak{p}_L$ is a trivial P-module there are isomorphisms of P-modules

$$L^{d-(k-i)} \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \cong L^{d-k} \otimes L^i \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \cong$$
$$L^{d-k} \otimes \mathrm{Sym}^i(\mathfrak{p}/\mathfrak{p}_L \otimes L) \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

for all $1 \leq i \leq k$. We get an injection

$$j : L^{d-k} \otimes \mathrm{Sym}^i(\mathfrak{p}/\mathfrak{p}_L \otimes L) \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \to L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

defined by

$$j(L^{d-k} \otimes \overline{y_1} \otimes L \cdots \overline{y_i} \otimes L \otimes \overline{x_1} \otimes L \cdots \overline{x_{k-i}} \otimes L) = L^{d-k} \otimes \overline{y_1} \otimes L \cdots \overline{y_i} \otimes L\overline{x_1} \otimes L \cdots \overline{x_{k-i}}.$$

The injection j gives rise to an injection

$$L^{d-(k-i)} \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \cong L^{d-k} \otimes \mathrm{Sym}^i(\mathfrak{p}/\mathfrak{p}_L \otimes L) \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \to^j$$
$$L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules for all $1 \leq i \leq k$. The exact sequence

$$0 \to \mathfrak{p}/\mathfrak{p}_L \to \mathfrak{g}/\mathfrak{p}_L \to \mathfrak{g}/\mathfrak{p} \to 0$$

gives rise to a filtration of P-modules

$$0 = F_{l+1} \subseteq F_l \subseteq \cdots \subseteq F_0 = \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

with quotients

$$F_i / F_{i+1} \cong L^i \otimes \mathrm{Sym}^{l-i}(\mathfrak{g}/\mathfrak{p} \otimes L).$$

Citation: Maakestad H (2015) Jet Bundles on Projective Space II. J Generalized Lie Theory Appl S2: 001. doi:10.4172/1736-4337.S2-001

Put $G_i = L^{d-1} \otimes F_i$. It follows

$$G_i = L^{d-(l-i)} \otimes \mathrm{Sym}^{l-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

There is an isomorphism

$$G_i / G_{i+1} \cong L^{d-(l-i)} \otimes \mathrm{Sym}^{l-i}(\mathfrak{g}/\mathfrak{p} \otimes L)$$

and claim 3.3.4 is proved.

We prove 3.3.5: Let $V = K\{e_0, .., e_n\}$ and $L = W = e_0$. It follows $P \subseteq G = \mathrm{SL}(V)$ is the group whose K-rational points are the following:

$$g = \begin{pmatrix} a & * \\ 0 & B \end{pmatrix}$$

with $a = \frac{1}{det(B)}$. Also B is an $n \times n$-matrix with coefficients in K. By definition the maps in the sequence are maps of P-modules. It follows $\mathfrak{p} = Lie(P)$ is the Lie algebra whose elements x are matrices on the following form:

$$x = \begin{pmatrix} -tr(B) & * \\ 0 & B \end{pmatrix}$$

where B is any $n \times n$-matrix with coefficients in K. The sub Lie algebra $\mathfrak{p}_L \subseteq \mathfrak{p}$ is the Lie algebra of matrixes $x \in \mathfrak{p}$ on the following form:

$$x = \begin{pmatrix} 0 & * \\ 0 & B \end{pmatrix}$$

where B is any $n \times n$-matrix with $tr(B) = 0$. Let $x_i \in \mathfrak{g}$ be the following element: Let the first column vector of x_i be the vector e_i and let the rest of the entries be such that $tr(x_i) = 0$. It follows $x_i \otimes e_0 \in \mathfrak{g} \otimes L$ and $x_i(e_0) = e_i$ hence the vertical map is surjective. One easily checks the sequence is exact and 3.3.5 is proved.

We get two P-modules: $\mathfrak{p}_L \subseteq \mathfrak{p}$ and $L^i = \mathrm{Sym}^i(\wedge^m W) \subseteq \mathrm{Sym}^i(\wedge^m V)$. We get for all $1 \leq k \leq d$ a P-module

$$L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

There is an injection of P-modules

$$i : L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L) \to \mathrm{Sym}^d(\wedge^m V)$$

defined by

$$i(L^{d-k} \otimes \overline{x_1} \otimes L \cdots \overline{x_k} \otimes L) = L^{d-k} x_1(L) \cdots x_k(L).$$

There are natural embeddings of P-modules

$$U_k(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(\wedge^m V)$$

and

$$L^{d-(k-1)} \otimes \mathrm{Sym}^{k-1}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(\wedge^m V).$$

Assume in the following $m = 1$ and $L = W$. It follows $\mathbb{G} = \mathcal{P}(V^*) = \mathcal{P}$ is projective n-space.

Proposition 3.4. *Let* $x_1 \cdots x_k(L^d) \in U_k(\mathfrak{g})L^d$. *The following formula holds:*

$$x_1 \cdots x_k(L^d) = \alpha L^{d-k} x_1(L) \cdots x_k(L) + \omega$$

where $\omega \in L^{d-(k-1)} \otimes \mathrm{Sym}^{k-1}(\mathfrak{g}/\mathfrak{p}_L \otimes L)$.

Proof. we prove the result by induction on k. Assume k=1 and let $x(L^d) \in U_1(\mathfrak{g})L^d$. It follows $x(L^d) = dL^{d-1}x(L) \in L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ and the claim holds for k =1. Assume the result is true for k. Hence

$$x_1 \cdots x_k(L^d) = \alpha L^{d-k} x_1(L) \cdots x_k(L) + \omega$$

with $\omega \in L^{d-(k-1)} \otimes \mathrm{Sym}^{k-1}(\mathfrak{g}/\mathfrak{p}_L \otimes L)$. Assume

$$\omega = \sum_i \alpha_i L^{d-(k-1)} x_1^i(L) \cdots x_{k-1}^i(L).$$

We get

$$x_0 x_1 \cdots x_k(L^d) = x_0(\alpha L^{d-k} x_1(L) \cdots x_k(L) + \omega) =$$

$$\alpha(d-k)L^{d-(k+1)} x_0(L) x_1(L) \cdots x_k(L) +$$

$$\sum_j \alpha L^{d-k} x_1(L) \cdots x_0(x_j(L)) \cdots x_k(L) +$$

$$\sum_i \alpha_i (d-(k-1)) L^{d-k} x_0(L) x_1^i(L) \cdots x_{k-1}^i(L) +$$

$$\sum_i \sum_l \alpha_i L^{d-(k-1)} x_1^i(L) \cdots x_0(x_l^i(L)) \cdots x_{k-1}^i(L).$$

Let $z_j(L) = x_0(x_j(L))$ and $z_l^i(L) = x\,(x_l^i(L))$. Such elements exist since $\mathfrak{g}/\mathfrak{p}_L \otimes L \cong V$ as P-module. Let

$$\omega = \sum_j \alpha L^{d-k} x_1(L) \cdots z_j(L) \cdots x_k(L) +$$

$$\sum_i \alpha_i (d-(k-1)) L^{d-k} x_0(L) \cdots x_1^i(L) \cdots x_{k-1}^i(L) +$$

$$\sum_i \sum_l \alpha_i L^{d-(k-1)} x_1^i(L) \cdots z_l^i(L) \cdots x_{k-1}^i(L).$$

it follows $\omega \in L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L)$. Moreover

$$x_0 x_1 \cdots x_k(L^d) = \tilde{\alpha} L^{d-(k+1)} x_0(L) \cdots x_k(L) + \omega$$

where $\tilde{\alpha} = (d-k)\alpha$. The Proposition is proved.

Theorem 3.5. *There is for all* $1 \leq l \leq d$ *an isomorphism*

$$\mathrm{U}_l(\mathfrak{g})L^d \cong L^{d\ l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules.

Proof. There are embeddings of P-modules

$$\mathrm{U}_l(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(V)$$

and

$$L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(V).$$

Recall from studies of Maakestad [1] it follows $dim_K(\mathrm{U}_l(\mathfrak{g})L^d) = \binom{l+n}{n}$ where $dim_k(V) = n$+1. Assume $z = x_1 \cdots x_l(L^d) \in \mathrm{U}_l(\mathfrak{g})L^d$. It follows from Proposition 3.4

$$z = \alpha L^{d-l} x_1(L) \cdots x_l(L) + \omega$$

where

$$\in L^{d-(l-1)}\mathrm{Sym}^{l-1}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

Since

$$\alpha L^{d-l} x_1(L) \cdots x_l(L) \in L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

it follows $z \in L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ Hence we get an inclusion of P-modules $\mathrm{U}_l(\mathfrak{g})L^d \subseteq L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$.

Since

$$dim_K(\mathrm{U}_l(\mathfrak{g})L^d) = dim_K(L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L))$$

the Theorem follows.

Corollary 3.6. *There is for all* $1 \leq l \leq d$ *an isomorphism*

$$\mathcal{P}_{\mathbb{P}}^l(\mathcal{O}_{\mathbb{P}}(d))(x) \cong (L^*)^{d-l} \otimes \mathrm{Sym}^1(V^*)$$

of P-modules.

Proof. There is by studies of Maakestad [1], Theorem 3.10 an isomorphism

$$\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d))(x)^* \cong \mathrm{U}_l(\mathfrak{g})L^d$$

of P-modules. From this isomorphism and Theorem 3.5 the Corollary follows since

$$(L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L))^* \cong (L^*)^{d-l} \otimes \mathrm{Sym}^l(V^*)$$

as P-modules.

Note: Corollary 3.6 is proved in literature of Maakestad [5] Theorem 2.4 using more elementary techniques.

Let $Y = \mathrm{Spec}(K)$ and $\pi : \mathcal{P}(V^*) \to Y$ be the structure morphism. Let $\mathcal{P} = \mathcal{P}(V^*)$. Since $\mathrm{Sym}^1(V^*)$ is a finite dimensional $\mathrm{SL}(V)$-module it follows it is a free $\mathcal{O}_Y$-module with an $\mathrm{SL}(V)$-linearization. It follows $\pi^*\mathrm{Sym}^1(V^*)$ is a locally free $\mathcal{O}\mathcal{P}$-module with an $\mathrm{SL}(V)$-linearization since π^* preserves the $\mathrm{SL}(V)$-linearization.

Proposition 3.7. *There is for all* $1 \le l \le d$ *an isomorphism*

$$\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d)) \cong \mathcal{O}_{\mathbb{P}}(d-l) \otimes \pi^*\mathrm{Sym}^l(V^*)$$

of locally free $\mathcal{O}\mathcal{P}$*-modules with an* $\mathrm{SL}(V)$*-linearization.*

Proof. Let $P \subseteq \mathrm{SL}(V)$ be the subgroup fixing the line $L \in V$ There is an exact equivalence of categories

$$\underline{mod}(P) \cong \underline{mod}^G(\mathcal{O}_{G/P}). \tag{3.7.1}$$

The P-module corresponding to $\mathcal{O}_{\mathbb{P}}(d-l) \otimes \pi^*\mathrm{Sym}^l(V^*)$ is $(L)^{d-l} \otimes \mathrm{Sym}^l(V^*)$. By the equivalence 3.7.1 and Corollary 3.6 we get an isomorphism

$$\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d)) \cong \mathcal{O}_{\mathbb{P}}(d-l) \otimes \pi^*\mathrm{Sym}^l(V^*)$$

of locally free sheaves with $\mathrm{SL}(V)$-linearization and the Proposition is proved.

We get a formula for the splitting type of $\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d))$ on projective space:

Corollary 3.8. *There is for all* $1 \le l \le d$ *an isomorphism*

$$\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}(d)) \cong \oplus^{\binom{n+l}{n}} \mathcal{O}_{\mathbb{P}}(d-l)$$

of locally free sheaves.

Proof. The P-modules $\mathrm{Sym}^1(V^*)$ corresponds to the free $\mathcal{O}\mathcal{P}$-module $\oplus^{\binom{n+l}{n}} \mathcal{O}_{\mathbb{P}}$. The Corollary now follows from Proposition 3.7.

Let $X = \mathbb{G}(m, m+n)$ and consider the P-modules

$$L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(\wedge^m V)$$

and

$$\mathrm{U}_1(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(\wedge^m V).$$

Proposition 3.9. *There is an isomorphism*

$$\mathrm{U}_1(\mathfrak{g})L^d \cong L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules.

Proof. Pick an element $x(L^d) = dL^{d-1}x(L) \in U_1(\mathfrak{g})L^d$. It follows $dL^{d-1}x(L) \in L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ hence there is an inclusion

$$\mathrm{U}_1(\mathfrak{g})L^d \subseteq L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

Let $L^{d-1}x(L) \in L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$. It follows

$$L^{d-1}x(L) = \frac{1}{d}x(L^d) \in \mathrm{U}_1(\mathfrak{g})L^d$$

hence there is an inclusion $L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ and the Proposition is proved.

Corollary 3.10. *There is an isomorphism*

$$\mathcal{P}^1_X(\mathcal{O}_X(d))(x)^* \cong L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules.

Proof. There is by studies of Maakestad [1], Theorem 3.10 an isomorphism

$$\mathcal{P}^1_X(\mathcal{O}_X(d))(x)^* \cong \mathrm{U}_1(\mathfrak{g})L^d$$

of P-modules. The Corollary follows from this fact and Proposition 5.1.

Note: By studies of Maakestad [11], Example 5.12 there is a double complex

$$\mathcal{O}_X(j) \otimes \mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*)$$

of sheaves on $\mathcal{P}(W^*)$ where $W = \mathrm{H}^0(X, \mathcal{O}_X(d))$ and $X = \mathbb{G}(m, m+n)$. This double complex might give rise to a resolution of the ideal sheaf of the l'th discriminant $D^l(\mathcal{O}_X(d)) \subseteq \mathbb{P}(W^*)$ of the line bundle $\mathcal{O}_X(d)$. By the literature of Maakestad, Theorem 5.2 it follows knowledge on the P-module structure of $\mathcal{P}^l_X(\mathcal{O}_X(d))$ gives information on the $\mathrm{SL}(V)$-module structure of the higher cohomology groups $\mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*)$ for all $i \ge 0$. This again gives information on the dimension $h^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*)$. We get a description of the locally free sheaf

$$\mathcal{O}_X(j) \otimes \mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*).$$

for all i, j.

Example 3.11. *Canonical filtration for the grassmannian* $\mathbb{G}(2,4)$.

Consider the example where $m = n = 2$ and $X = \mathbb{G}(2,4)$. We get two inclusions

$$L^{d-2} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(\wedge^2 V)$$

and

$$\mathrm{U}_2(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(\wedge^2 V).$$

We may choose a basis for $\mathfrak{p} \subseteq \mathfrak{g}$ on the following form:

$$\mathfrak{p} = \mathfrak{p}_L \oplus L_x$$

where L_x is the line spanned by the following vector x:

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0. \end{pmatrix}$$

Let $\mathfrak{n} \subseteq \mathfrak{g}$ be the sub Lie algebra spanned by the following vectors:

$$x_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Citation: Maakestad H (2015) Jet Bundles on Projective Space II. J Generalized Lie Theory Appl S2: 001. doi:10.4172/1736-4337.S2-001

$$x_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$x_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $\tilde{\mathfrak{n}}$ be the vector space spanned by the vectors x_1, x_2, x_4, x_4 and x. It follows $\mathrm{U}_2(\mathfrak{g})L^d = \mathrm{U}_2(\tilde{\mathfrak{n}})L^d \subseteq \mathrm{Sym}^d(\wedge^2 V)$. The vector space V has a basus e_1, e_2, e_3 and e_4. The vector space W has basis e_1, e_2. It follows $\wedge^2 W$ has a basis given by $e_1 \wedge e_2 = e[12]$ and $\wedge^2 V$ has basis given by $e[12]$, $e[13]$, $e[14]$, $e[23]$, $e[24]$, $e[34]$. By definition $L = e[12]$. We get the following calculation:

$$x_1(L) = -e[23], x_2(L) = e[13], x_3(L) = -e[24]$$

$$x_4(L) = e[14], x(L) = e[12].$$

A basis for the P-module $L^{d-2} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ are the following vectors:

$$L^{d-2}x(L)x(L) = L^{d-2}e[12]^2$$

$$L^{d-2}x_2(L)x(L) = L^{d-2}e[12]e[13]$$

$$L^{d-2}x_4(L)x(L) = L^{d-2}e[12]e[14]$$

$$L^{d-2}x_1(L)x(L) = -L^{d-2}e[12]e[23]$$

$$L^{d-2}x_3(L)x(L) = -L^{d-2}e[12]e[24]$$

$$L^{d-2}x_2(L)x_2(L) = L^{d-2}e[13]^2$$

$$L^{d-2}x_2(L)x_4(L) = L^{d-2}e[13]e[14]$$

$$L^{d-2}x_1(L)x_2(L) = -L^{d-2}e[13]e[23]$$

$$L^{d-2}x_2(L)x_3(L) = -L^{d-2}e[13]e[24]$$

$$L^{d-2}x_4(L)x_4(L) = -L^{d-2}e[14]^2$$

$$L^{d-2}x_1(L)x_4(L) = -L^{d-2}e[14]e[23]$$

$$L^{d-2}x_3(L)x_4(L) = -L^{d-2}e[14]e[24]$$

$$L^{d-2}x_1(L)x_1(L) = L^{d-2}e[23]^2 e$$

$L^{d-2}x_1$ [illegible] $L^{d-2}e$ e

$$L^{d-2}x_3(L)x_3(L) = L^{d-2}e[24]^2$$

Let $a = d(d-1)$. A basis for the P-module $\mathrm{U}_2(\mathfrak{g})L^d = \mathrm{U}_2(\tilde{\mathfrak{n}})L^d$ are the following vectors:

$$x^2(L^d) = L^{d-2}e[12]^2$$

$$x_2x(L^d) = aL^{d-2}e[12]e[13] + dL^{d-1}e[13]$$

$$x_4x(L^d) = aL^{d-2}e[12]e[14] + dL^{d-1}e[14]$$

$$x_1x(L^d) = aL^{d-2}e[12]e[23] - dL^{d-1}e[23]$$

$$x_3x(L^d) = aL^{d-2}e[12]e[24] - dL^{d-1}e[24]$$

$$x_2^2(L^d) = aL^{d-2}e[13]^2$$

$$x_2x_4(L^d) = aL^{d-2}e[13]e[14]$$

$$x_1x_2(L^d) = aL^{d-2}e[13]e[23]$$

$$x_2x_3(l^D) = -aL^{d-2}e[13]e[24] - dL^{d-1}e[34]$$

$$x_4^2(L^d) = L^{d-2}e[14]^2$$

$$x_1x_4(L^d) = -aL^{d-2}e[14]e[23] + dL^{d-1}e[34]$$

$$x_3x_4(L^d) = -aL^{d-2}e[14]e[24]$$

$$x_1^2(L^d) = aL^{d-2}e[23]^2$$

$$x_1x_3(L^d) = aL^{d-2}e[23]e[24]$$

$$x_3^2(L^d) = aL^{d-2}e[24]^2.$$

In the case where $W \subseteq V$ have dimensions m and $m + n$ we get embeddings of P-modules

$$\mathrm{U}_l(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(\wedge^m V)$$

and

$$L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(\wedge^m V).$$

There is no equality

$$\mathrm{U}_l(\mathfrak{g})L^d = L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules as submodules of $\mathrm{Sym}^d(\wedge^m V)$ in general as Example 3.11 shows.

Since $\mathrm{U}_l(\mathfrak{g})L^d$ and $L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ by Theorem 3.5 and Proposition 3.3 are isomorphic when $m = 1$ and $1 \leq l \leq d$, have the same dimension over K and both have natural filtrations of P-modules we may conjecture they are isomorphic as P-modules for all $m,n \geq 1$. Note: There is a canonical line $L^d \in \mathrm{U}_l(\mathfrak{g})L^d$ for all l. There is similarly a canonical line

$$L^d \cong L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{p}/\mathfrak{p}_L \otimes L) \in L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

Hence the two P-modules $\mathrm{U}_l(\mathfrak{g})L^d$ and $L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ look similar.

In general the $\mathrm{SL}(V)$-module $\mathrm{Sym}^d(\wedge^m V)$ decompose

$$\mathrm{Sym}^d(\wedge^m V) \cong \oplus_i V_{\lambda_i}^{a_i}$$

where V_{λ_i} are irreducible $\mathrm{SL}(V)$-modules and $a_i \geq 1$ are integers (Proposition 5.4 for the situation of $\mathbb{G}(2,4)$. One may ask if there is a non-trivial automorphism

$$\phi \in \mathrm{Aut}_{\mathrm{SL}(V)}(\mathrm{Sym}^d(\wedge^m V))$$

with the property that the morphism

$$\phi : \mathrm{Sym}^d(\wedge^m V) \to \mathrm{Sym}^d(\wedge^m V)$$

induce an isomorphism

$$\tilde{\phi} : L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L) \to \mathrm{U}_l(\mathfrak{g})L^d$$

of P-modules. In general the $\mathrm{SL}(V)$-module $\mathrm{Sym}^d(\wedge^m V)$ has lots of automorphisms. When $m = 2$ and $dim_k(V) = 4$ it follows by Corollary 5.4 there is for every $d \geq 1$ an equality

$$\mathrm{Aut}_{\mathrm{SL}(V)}(\mathrm{Sym}^d(\wedge^2 V)) = \prod_{i=0}^{l} K^*$$

where $l = k$ if $d = 2k$ or $d = 2k + 1$. For $m = n = 2$ the $\mathrm{SL}(V)$-module $\mathrm{Sym}^d(\wedge^m V)$ is by Proposition 5.4 multiplicity free. The module $\mathrm{Sym}^d(\wedge^m K^{m+n})$ is not multiplicity free in general when $m, n > 2$.

Jet Bundles and Incidence Complexes on the Projective Line

In this section we construct a resolution by locally free sheaves of the ideal sheaf of the l'th incidence scheme $I^l(\mathcal{O}_{\mathbb{P}}(d)) \subseteq \mathbb{P}(W^*) \times \mathbb{P}$. Here $\mathcal{OP}(d)$ is an invertible sheaf on the projective line $\mathcal{P} = \mathcal{P}^1$ and $W = \mathrm{H}^0(\mathbb{P}, \mathcal{O}_p(d))$. There is on $Y = \mathcal{P}(W^*) \times \mathcal{P}^1$ a morphism $\varphi(\mathcal{O}(d))$ of locally free sheaves

$$\phi(\mathcal{O}(d)) : \mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \to \mathcal{P}^l(\mathcal{O}(d))_Y$$

Its zero scheme $Z(\phi(\mathcal{O}(d))) = I^l(\mathcal{O}(d)) \subseteq Y$ is the l'th incidence scheme of $\mathcal{O}(d)$. The Koszul complex of the morphism $\varphi(\mathcal{O}(d))$

$$0 \to \wedge^l \mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^* \to \cdots \to \wedge^2 \mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^* \to$$

$$\mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^* \to \mathcal{O}_Y \to \mathcal{O}_{I^l(\mathcal{O}(d))} \to 0$$

- called the *incidence complex of* $\mathcal{O}(d)$ - is a resolution of the ideal sheaf of $I^l(\mathcal{O}(d))$. This follows from the fact that the ideal sheaf of $I^l(\mathcal{O}(d))$ is locally generated by a regular sequence. We also calculate the higher direct images of the terms

$$\mathcal{O}(-j)_Y \otimes \wedge^j \mathcal{P}^l(\mathcal{O}(d))_Y^*$$

appearing in the incidence complex.

The aim of the construction is to use it to construct a resolution of the ideal sheaf of the discriminant $D^l(\mathcal{O}(d))$ where $\mathcal{O}(d)$ is a line bundle on projective space or a grassmannian.

Example 4.1. *The Koszul complex of a map of locally free modules.*

Let A be an arbitrary commutative ring with unit and let $\varphi : E \to F$ be a map A-modules.

Define the following map:

$$d^0 : E \otimes_A F^* \to A$$

by

$$d^0(x \otimes f) = f(\phi(x)).$$

Let I A be the image of d^1. We let I_φ be the *ideal of* φ. Define the following map

$$d^p : \wedge^p E \otimes F^* \to \wedge^{p-1} E \otimes F^*$$

by

$$d^p(x_1 \otimes f_1 \wedge \cdots \wedge x_p \otimes f_p) = \sum_{r=1}^{p} (-1)^{r-1} f_r(\phi(x_r)) x_1 \otimes f_1 \wedge \cdots \wedge \widetilde{x_r \otimes f_r} \wedge \cdots \wedge x_p \otimes f_p.$$

Lemma 4.2. *The following holds for all* $p \geq 1$: $d^p \circ d^{p-1} = 0$.

Proof. We get

$$d^{p-1} d^p(x_1 \otimes f_1 \wedge \cdots \wedge x_p \otimes f_p) =$$

$$\sum_{r=1}^{p} (-1)^{r-1} f_r(\phi(x_r))$$

$$\sum_{l \neq r} (-1)^{l-1} f_l(\phi(x_l)) x_1 \otimes f_1 \wedge \cdots \wedge \widetilde{x_l \otimes f_l} \wedge \cdots \wedge \widetilde{x_r \otimes f_r} \wedge \cdots \wedge x_p \otimes f_p = 0$$

and the claim of the Lemma follows.

Assume E, F are locally free of finite rank and let $r = rk(E \otimes F^*)$. We get a complex of locally free A-modules

$$0 \to \wedge^r E \otimes F^* \to \cdots \to \wedge^2 E \otimes F^* \to E \otimes F^* \to A \to A / I_\phi \to 0$$

called the *Koszul complex of the map* φ

Example 4.3. *The Koszul complex of a regular sequence.*

Let $\underline{x} = \{x_1, .., x_n\}$ be a regular sequence of elements in A and let $E = Ae$ be the free A-module on the element e. Let $F = A\{e_1, .., e_n\}$ be a free rank n module on $e_1, .., e_n$. Let $y_i = e_i^*$. Define

$$\varphi : E \to F$$

by

$$\phi(e) = x_1 e_1 + \cdots + x_n e_n.$$

Let $e \otimes y_i = z_i$. It follows

$$d^p : \wedge^p E \otimes F^* \to \wedge^{p-1} E \otimes F^*$$

looks as follows:

$$d^p(z_{i_1} \wedge \cdots \wedge z_{i_p}) =$$

$$\sum_{r=1}^{p} (-1)^{p-1} y_{i_r}(\phi(e)) z_{i_1} \wedge \cdots \wedge \widetilde{z_{i_r}} \wedge \cdots \wedge z_{i_p} =$$

$$\sum_{r=1}^{p} (-1)^{r-1} x_{i_r} z_{i_1} \wedge \cdots \wedge \widetilde{z_{i_r}} \wedge \cdots \wedge z_{i_p}.$$

Hence the complex $\wedge^\bullet E \otimes F^*$ equals the Koszul complex $K_\bullet(\underline{x})$ of the regular sequence $\underline{x}$. It is an exact complex since $\underline{x}$ is a regular sequence.

Example 4.4. *The Koszul complex of a morphism of locally free sheaves.*

The construction of the differential in the Koszul complex of a map of modules is intrinsic, hence we may generalize to morphisms of locally free sheaves. Let Y be an arbitrary scheme and let $\varphi : \mathcal{E} \to \mathcal{F}$ be a map of locally free $\mathcal{O}_Y$-modules. Let

$$d^0 : \mathcal{E} \otimes \mathcal{F}^* \to \mathcal{O}_Y$$

be defined locally by

$$d^0(s \otimes v) = v(\phi(s)).$$

Let $\mathcal{I}_\phi = Im(d^0) \subseteq \mathcal{O}_Y$ be the ideal sheaf defined by d^1. Since $\mathcal{I}_\varphi$ is quasi coherent sheaf of ideals it follows the ideal sheaf $\mathcal{I}_\varphi$ corresponds to a subscheme $Z(\varphi) \subseteq Y$ - the *zero scheme of* φ. Let $U \subseteq Y$ be an open subset and define the following map:

$$d^p : \wedge^p(\mathcal{E} \otimes \mathcal{F}^*)(U) \to \wedge^{p-1}(\mathcal{E} \otimes \mathcal{F}^*)(U)$$

by

$$d^p(s_1 \otimes v_1 \wedge \cdots \wedge s_p \otimes v_p) = \sum_{r=1}^{p} (-1)^{r-1} v_r(\phi(s_r)) s_1 \otimes v_1 \wedge \cdots \wedge \widetilde{s_r \otimes v_r} \wedge \cdots \wedge s_p \otimes v_p.$$

This gives a well defined map of locally free sheaves since we have not chosen a basis for the module $\wedge^p(\mathcal{E} \otimes \mathcal{F}^*)(U)$ to give a definition. By Lemma 4.2 it follows $d^p \circ d^{p+1} = 0$ for all $p \geq 1$ hence we get a complex of locally free sheaves. The sequence of maps of locally free sheaves

$$0 \to \wedge^r \mathcal{E} \otimes \mathcal{F}^* \to \cdots \to \wedge^2 \mathcal{E} \otimes \mathcal{F}^* \to \mathcal{E} \otimes \mathcal{F}^* \to \mathcal{O}_Y \to \mathcal{O}_{Z(\phi)} \to 0$$

is called the *Koszul complex of* ϕ. Here $r = rk(\mathcal{E} \otimes \mathcal{F})$.

Example 4.5. *Koszul complexes and local complete intersections.*

Assume $\phi : \mathcal{L} \to \mathcal{F}$ is a map of locally free $\mathcal{O}_Y$-modules where $\mathcal{L}$ is a line bundle. Let $Z(\phi) \subseteq Y$ be the subscheme defined by ϕ - the zero scheme of ϕ. Let $r = rk(\mathcal{F})$. Choose an open affine cover U_i of Y where $\mathcal{F}$ and $\mathcal{L}$ trivialize, i.e

$$\mathcal{F}(U_i) = \mathcal{O}(U_i)\{f_{i1}, .., f_{ir}\}$$

and

$$\mathcal{L}(U_i) = \mathcal{O}(U_i) e_i.$$

Let $\mathcal{O}(U_i) = A_i$, $L_i = \mathcal{L}(U_i)$ and $F_i = \mathcal{F}(U_i)$. Assume the image

$$\phi(U_i): L_i \to F_i$$

has

$$\phi(U_i)(e_i) = x_{i1}f_{i1} + \cdots + x_{ir}f_{ir}$$

where $\{x_{i1},..,x_{ir}\} \subseteq A_i$ is a regular sequence. Let $I_i = \underline{x_i} = \{x_{i1},..,x_{ir}\}$. It follows from Example 4.3 the Koszul complex

$$0 \to \wedge^r(L_i \otimes F_i^*) \to \cdots \to \wedge^2(L_i \otimes F_i^*) \to L_i \otimes F_i^*$$

$$\to A_i \to A_i / I_i \to 0$$

is a resolution of the ideal I_i since I_i is generated by a regular sequence. The complex $\wedge^{\bullet} L_i \otimes F_i^*$ is isomorphic to the Koszul complex $K_{\bullet}(\underline{x_i})$ on the regular sequence $\underline{x_i}$. It follows the global complex

$$0 \to \mathcal{L}^{\otimes r} \wedge^r \mathcal{F}^* \to \cdots \to \mathcal{L}^{\otimes 2} \wedge^2 \mathcal{F}^* \to \mathcal{L} \otimes \mathcal{F}^* \to \mathcal{O}_Y \to \mathcal{O}_{Z(\phi)} \to 0$$

is a resolution of the ideal sheaf $\mathcal{I}_{Z(\varphi)}$ of $Z(\varphi) \subseteq Y$ since it is locally isomorphic to the Koszul complex $K_{\bullet}(\underline{x_i})$ for all i.

Since the ideal I_i is generated by a regular sequence of length r it follows $dim(A_i / I_i) = dim(A_i) - r$. If Y is irreducible of dimension d it follows $Z(\varphi) \subseteq Y$ is a local complete intersection of dimension $d - r$.

Example 4.6. *The incidence complex of $\mathcal{O}(d)$ on the projective line.*

Let $\mathbb{P} = \mathbb{P}^1_K$ where K is a field of characteristic zero and let $\mathcal{O}(d) \in \mathrm{Pic}(\mathcal{P}) = \mathrm{Z}$ be a line bundle where $d \in \mathrm{Z}$. Let

$$W = \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d)) = K\{e_0,..,e_d\}$$

where $e_i = x_0^{d-i}x_1^i$. Let $y_i = e_i^*$. Let $Y = \mathcal{P}(W^*) \times \mathcal{P}$ and consider the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{p} & \mathbb{P} \\ \downarrow q & & \downarrow \pi \\ \mathbb{P}(W^*) & \xrightarrow{\pi} & \mathrm{Spec}(K) \end{array}.$$

There is a sequence of locally free $\mathcal{O}_Y$-modules

$$\mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \to \mathrm{H}^0(\mathbb{P}.\mathcal{O}(d)) \otimes \mathcal{O}_Y \xrightarrow{T^l_Y} \mathcal{P}^l(\mathcal{O}(d))_Y$$

and let φ $(\mathcal{O}(d))$ be the composed map

$$\phi(\mathcal{O}(d)) : \mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \to \mathcal{P}^l(\mathcal{O}(d))_Y. \qquad (12)$$

It follows by studies of Maakestad [11], the zero scheme $Z(\varphi(\mathcal{O}(d)))$ equals the incidence scheme $I^l(\mathcal{O}(d))$ of the line bundle $\mathcal{O}(d)$. By definition $\mathbb{P}(W^*) = \mathrm{Proj}(K[y_0,..,y_d])$ where $y_i = e_i^*$. It has an open cover on the following form: $D(y_i) = \mathrm{Spec}(K[u_0,..,u_d])$ where we let $u_j = \frac{y_j}{y_i}$. Let $y_j / y_j = 1$. Let

$$F(t) = u_0 + u_1 t + \cdots + u_d t^d \in K[u_0,..,u_d,t].$$

Restrict the map 4.6.1 to the open set $U_{i0} = D(y_i) \times D(x_0) \subseteq Y$. We get the following two maps of modules:

$$\alpha : \mathcal{O}_{\mathbb{P}(W^*)}(-1)|_{U_{i0}} \to \mathcal{O}_{U_{i0}} \otimes \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d))$$

$$\alpha : K[y_i,t]\frac{1}{y_i} \to K[u_i,t] \otimes K\{e_0,..,e_d\}$$

defined by

$$\alpha(1/y_i) = \sum_{k=0}^d u_k \otimes e_k = \sum_{k=0}^d u_k \otimes x_0^{d-k}x_1^k = \sum_{k=0}^d u_k \otimes t^k x_0^d.$$

We get the map

$$T^l_{U_{i0}} : \mathcal{O}_{U_{i0}} \otimes \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d)) \to \mathcal{P}^l(\mathcal{O}(d))|_{U_{i0}}$$

defined by

$$T^l(1 \otimes x_0^{d-i}x_1^i) = T^l(1 \otimes t^i x_0^d) = (t + dt)^i \otimes x_0^d.$$

The composed map

$$\phi(\mathcal{O}(d))_{U_{i0}} : K[u_i,t]\frac{1}{y_i} \to K[u_i,t]\{dt^j \otimes x_0^d\}$$

is the map

$$\phi(\mathcal{O}(d))(\frac{1}{y_i}) = \sum_{k=0}^d u_k (t + dt)^k \otimes x_0^d =$$

$$\sum_{k=0}^l \frac{F^{(k)}(t)}{k!} dt^k \otimes x_0^d \in K[u_i,t]\{1 \otimes x_0^d,..,dt^l \otimes x_0^d\}.$$

Let $U_{i1} = D(y_i) \times D(x_1) \subseteq Y$ and let $\frac{x_0}{x_1} = s$. Let

$$G(s) = u_d + u_{d-1}s + u_{d-2}s^2 + \cdots + u_0 s^d \in K[u_0,..,u_d,s].$$

Restrict the map 4.6.1 to the open set U_{i1}

We get the following two maps of modules:

$$\alpha : \mathcal{O}_{\mathbb{P}(W^*)}(-1)|_{U_{i1}} \to \mathcal{O}_{U_{i1}} \otimes \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d))$$

$$\alpha : K[y_i,s]\frac{1}{y_i} \to K[u_i,s] \otimes K\{e_0,..,e_d\}$$

defined by

$$\alpha(1/y_i) = \sum_{k=0}^d u_k \otimes e_k = \sum_{k=0}^d u_k \otimes x_0^{d-k}x_1^k = \sum_{k=0}^d u_k \otimes s^{d-k}x_1^d.$$

We get the map

$$T^l_{U_{i1}} : \mathcal{O}_{U_{i1}} \otimes \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d)) \to \mathcal{P}^l(\mathcal{O}(d))|_{U_{i1}}$$

defined by

$$T^l(1 \otimes x_0^{d-i}x_1^i) = T^l(1 \otimes s^{d-i}x_1^d) = (s + ds)^{d-i} \otimes x_1^d.$$

The composed map

$$\phi(\mathcal{O}(d))_{U_{i1}} : K[u_i,s]\frac{1}{y_i} \to K[u_i,s]\{ds^j \otimes x_1^d\}$$

is the map

$$\phi(\mathcal{O}(d))(\frac{1}{y_i}) = \sum_{k=0}^d u_{d-k}(s + ds)^k \otimes x_1^d =$$

$$\sum_{k=0}^l \frac{G^{(k)}(s)}{k!} ds^k \otimes x_1^d \in K[u_i,s]\{1 \otimes x_1^d,..,ds^l \otimes x_1^d\}.$$

It follows the ideal sheaf $\mathcal{I}_{I^l(\mathcal{O}(d))}$ of $I^l(\mathcal{O}(d))$ is generated by

$$\{\frac{F^{(l)}(t)}{l!}, \frac{F^{(l-1)}(t)}{(l-1)!},..,F(t)\}$$

on U_{i0} and by

$$\{\frac{G^{(l)}(s)}{l!}, \frac{G^{(l-1)}(s)}{(l-1)!},..,G(s)\}$$

on U_{i1}. Let $z_i = \frac{F^{(i)}(t)}{(i)!}$ and $w_i = \frac{G^{(i)}(s)}{(i)!}$ for $i = 0,..,l$.

Lemma 4.7. *Assume B is a commutative ring of characteristic zero and let*

$$f(t) = a_0 + a_1 t + \cdots + a_d t^d \in B[t]$$

be an arbitrary degree d polynomial with $a_d \neq 0$. Let $f^{(i)}(t)$ denote the formal derivative with respect to t. It follows

$$\frac{f^{(k)}(t)}{l!} = \sum_{i=k}^{d} \binom{i}{k} a_i t^{i-k}.$$

Proof. The proof is by induction. It is clearly true for $l = 1$. Assume it is true for $l > 1$. Consider $k = l + 1$. We get

$$\frac{f^{(l+1)}(t)}{(l+1)!} = \frac{1}{l+1}\frac{\partial}{\partial_t}\frac{f^{(l)(t)}}{l!} =$$

$$\frac{1}{l+1}(\binom{l+1}{l}a_{l+1} + \binom{l+2}{l}2a_{l+2}t + \cdots + \binom{d}{l}(d-l)a_d t^{d-(l+1)}) =$$

$$\binom{l+1}{l+1}a_{l+1} + \binom{l+2}{l+1}a_{l+2}t + \cdots + \binom{d}{l+1}a_d t^{d-(l+1)} =$$

$$\sum_{i=l+1}^{d}\binom{i}{l+1}a_i t^{i-(l+1)}$$

and the claim of the Lemma follows.

Lemma 4.8. *The sequence* $\{z_l, .., z_0\}$ *is a regular sequence in* $K[u_i, t]$. *The sequence* $\{w_l, .., w_0\}$ *is a regular sequence in* $K[u_i, s]$.

Proof. Let $z_i = \frac{F^{(i)}(t)}{i!}$ and $w_j = \frac{G^{(j)}}{j!}$. Assume $l < i$ and consider the sequence $z_l, z_{l-1}, .., z_0 \subseteq A[t] = K[u_0, .., u_d][t]$. Since $A[t]$ is a domain it follows z_l is a non zero divisor in $A[t]$. We see from Lemma 4.7

$$A[t]/w_l \cong K[u_0, .., u_{l-1}, u_{l+1}, .., u_d, t]$$

which is a domain, hence w_{l-1} is a non zero divisor in $A[t] / w_l$. By induction it follows $z_l, .., z_0$ is a regular sequence in $A[t]$. Assume $i \le l$. It follows the sequence $z_l, .., z_{i+1}$ is a regular sequence in $A[t]$. We see from Lemma 4.7 z_l is non zero in

$$A[t]/(z_l, .., z_{i+1}) = K[u_0, .., u_i, u_{l+1}, .., u_d, t]$$

and $K[u_0, .., u_i, u_{l+1}, .., u_d, t]$ is a domain. It follows z_i is a non zero divisor in $A[t] / (z_l, .., z_{i+1})$. It follows $z_l, .., z_0$ is a regular sequence in $A[t]$ and the claim follows. A similar argument proves $w_l, .., w_0$ is a regular sequence in $A[s]$ and the Lemma is proved.

One may prove using similar methods for any permutation $\sigma \in S_{l+1}$ the sequences

$$z_{(l)}, .., z_{\sigma(0)}$$

and

$$w_{(l)}, .., w_{\sigma(0)}$$

are regular sequences.

It follows the ideal sheaf $\mathcal{I}_{I^l(\mathcal{O}(d))}$ is locally generated by a regular sequence.

The morphism

$$\phi(\mathcal{O}(d)) : \mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \to \mathcal{P}^l(\mathcal{O}(d))_Y$$

gives by Example 4.3 rise to a Koszul complex

$$\wedge^{\bullet}\mathcal{O}_{\mathbb{P}(W^*)}(-1) \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^*$$

of locally free sheaves of $Y = \mathcal{P}(W^*) \times \mathcal{P}^1$.

Definition 4.9. Let the complex

$$0 \to \wedge^l \mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^* \to \cdots \to \wedge^2 \mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^* \to \quad (4.9.1)$$
$$\mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^* \to \mathcal{O}_Y \to \mathcal{O}_{I^l(\mathcal{O}(d))} \to 0$$

be the *incidence complex of* $\mathcal{O}(d)$.

Since the ideal sheaf of $I^l(\mathcal{O}(d))$ by the discussion above is locally generated by a regular sequence it follows from Example 4.3 the complex 4.9.1 is a resolution.

In framework of Maakestad [5], Theorem 5.10 one calculates the higer direct images

$$\mathrm{R}^i q_*(\wedge^j \mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^*)$$

for all i, j. We get the following calculations:

Let $V = K\{e_0, e_1\}$ and $\mathcal{P} = \mathcal{P}(V^*)$. Let $W = \mathrm{H}^0(\mathcal{P}, \mathcal{O}(d)) = \mathrm{Sym}^d(V^*)$ and consider the diagram

$$\begin{tikzcd} Y = \mathbb{P}(W^*) \times \mathbb{P} \arrow[r, "p"] \arrow[d, "q"] & \mathbb{P} \arrow[d, "\pi"] \\ \mathbb{P}(W^*) \arrow[r, "\pi"] & \mathrm{Spec}(K) \end{tikzcd}.$$

By the results of this paper it follows there is an isomorphism

$$\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}(d)) \cong \mathcal{O}_{\mathbb{P}}(d-l) \otimes \pi^* \mathrm{Sym}^l(V^*)$$

a sheaves with an $\mathrm{SL}(V)$-linearization. We get

$$\wedge^j \mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d)) \cong \mathcal{O}_{\mathbb{P}}(j(d-l)) \otimes \pi^* \wedge^j \mathrm{Sym}^l(V^*).$$

By the equivariant projection formula for higher direct images we get

$$\mathrm{R}^i q_*(\wedge^j \mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^*) \cong \mathcal{O}_{\mathbb{P}(W^*)}(-j) \otimes \mathrm{H}^i(\mathbb{P}, \wedge^j \mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d))^*).$$

Let

$\pi : \mathcal{P} \to \mathrm{Spec}(K)$.

It follows

$$\wedge^j \mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d))^* \cong \mathcal{O}_{\mathbb{P}}(j(l-d)) \otimes \pi^* \wedge^j \mathrm{Sym}^l(V^*).$$

We get

$$\mathrm{H}^i(\mathbb{P}, \wedge^j \mathcal{P}^l(\mathcal{O}(d))^*) \cong \mathrm{R}^i \pi_*(\pi^*(\wedge^j \mathrm{Sym}^l(V)) \otimes \mathcal{O}_{\mathbb{P}}(j(l-d))) \cong$$

$$\wedge^j(\mathrm{Sym}^l(V)) \otimes \mathrm{H}^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(j(l-d))).$$

We get the following Theorem:

Theorem 4.10. *The following holds:*

$$\mathrm{R}^i p_*(\mathcal{O}(-j) \otimes \wedge^j \mathcal{P}^l(\mathcal{O}(d))^*) = 0 \text{ if } i = 0 \text{ or } i = 1 \text{ and } j(d-l) < 2.$$

$$\mathrm{R}^1 p_*(\mathcal{O}(-j) \otimes \wedge^j \mathcal{P}^l(\mathcal{O}(d))^*) = \mathcal{O}(-j) \otimes \mathrm{Sym}^{j(d-l)-2}(V) \otimes \wedge^j \mathrm{Sym}^l(V)$$

if $j(d - l) \ge 2$.

Proof. The proof follows from the calculation of the equivariant cohomology of line bundles on projective space [13].

Hence we have complete control on the sheaf

$$\mathrm{R}^i q_*(\wedge^j \mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^*)$$

on the projective line and projective space for all i, j. Using the techniques introduced in this paper one may describe resolutions of incidence schemes $I^l(\mathcal{O}(d))$ on more general grassmannians and flag varieties. The hope is we may be able to construct resolutions of the ideal sheaf of $D^l(\mathcal{O}(d))$ using indicence resolutions in a more general situation.

Note: In literature of Lascoux [12] resolutions of ideal sheaves of determinantal schemes are studied and much is known on such resolutions. In studies of Maakestad [11] it is proved $D^l(\mathcal{O}(d)$ is a determinantal scheme for any $d \geq 2$ on the projective line $\mathcal{PP}^1$. Assume $\mathcal{L} \in \mathrm{Pic}^G(G/P)$ is a G-linearized linebundle, G a semi simple linear algebraic group and P a parabolic subgroup. If one can prove $D^l(\mathcal{L})$ is a determinantal scheme we get two approaches to the study of resolutions of ideal sheaves of discriminants: One using jet bundles and incidence schemes, another one using determinantal schemes.

Appendix A: Automorphisms of Representations

Let $W \subseteq V$ be vectorspaces of dimension two and four over the field K. Consider the subgroup $P \subseteq G = \mathrm{SL}(V)$ where P is the parabolic subgroup of elements fixing W. It follows $\pi : G \to G/P = \mathbb{G}(2,4)$ is a principal P-bundle. Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{p} = Lie(P)$ be the Lie algebras of G and P. In this section we study the decomposition into irreducibles and automorphisms of some G-modules. We also study some P_{semi}-modules where P_{semi} is the semi-simplification of P. It follows P_{semi} equals $\mathrm{SL}(2) \times \mathrm{SL}(2)$. Since $\mathfrak{p} \subseteq \mathfrak{g}$ is a P-sub module it follows the quotient $\mathfrak{g}/\mathfrak{p}$ is a P-module hence a P_{semi} module. We may apply the theory of highest weights since $P_{semi} = \mathrm{SL}(2) \times \mathrm{SL}(2)$ is a semi simple algebraic group.

Proposition 5.1. *The following hold: There is an isomorphism of* $\mathrm{SL}(2) \times \mathrm{SL}(2)$*-modules*

$$\mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}) = \oplus_{i=0}^n \mathrm{Sym}^{2i+m}(W^*) \otimes \mathrm{Sym}^{2i+m}(V/W). \tag{5.1.1}$$

for all $k \geq 1$. *Here* $(n,m) = (-,0)$ *if* $k = 2n$ *and* $(n,m) = (\frac{k-1}{2},1)$ *if* $k = 2n + 1$.

Proof. Recall the canonical isomorphism from Lemma 2.4

$$\mathfrak{g}/\mathfrak{p} \cong \mathrm{Hom}(W, V/W) \cong W^* \otimes V/W$$

of P-modules. It follows

$$\mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}) \cong \mathrm{Sym}^k(W^* \otimes V/W)$$

and its decomposition into irreducible $\mathrm{SL}(2) \times \mathrm{SL}(2)$-modules can be done using well known formulas [14]. Alternatively one may compute its highest weight vectors and highest weights explicitly using the construction from Section 5.

Let $i : G/P \to \mathcal{P}(\wedge^2 V^*) = \mathcal{P}$ be the Plucker embedding and let $\mathcal{O}_{G/P}(1) = i^*\mathcal{OP}(1)$ be tautological line bundle on G/P and let $\mathcal{O}_{G/P}(d) = \mathcal{O}_{G/P}(1)\otimes^d$. It follows from the Borel-Weil-Bott Theorem [16] $\mathrm{H}^0(\mathbb{G},\mathcal{O}\mathbb{G}(d))$ is an irreducible $\mathrm{SL}(V)$-module. Let V have basis e_1, e_2, e_3, e_4 and let $\wedge^2 V$ have basis e_{ij} for $1 \leq i \leq j \leq 4$, with $e_{ij} = e_i \wedge e_j$. Consider the element $f \in \mathrm{Sym}^2(\wedge^2 V)$ where

$$f = e_{12}e_{34} - e_{13}e_{24} + e_{14}e_{23}.$$

One checks f is a highest weight vector for $\mathrm{SL}(V)$ with highest weight 0, hence it defines the unique trivial character of $\mathrm{SL}(V)$. Its dual

$$f^* = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} \in \mathrm{Sym}^2(\wedge^2 V^*)$$

is the defining equation for $\mathbb{G} = G/P$ as closed subscheme of $\mathcal{P}(\wedge^2 V^*)$.

Proposition 5.2. *The following hold: there is an isomorphism of* $\mathrm{SL}(V)$*-modules*

$$\mathrm{Sym}^d(\wedge^2 V) = \oplus_{i=0}^l \mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}}(d-2i))^*, \tag{5.2.1}$$

where $l = k$ *if* $d = 2k$ *or* $d = 2k + 1$.

Proof. The result is proved using the theory of highest weights. There is a split exact sequence of $\mathrm{SL}(V)$-modules

$$0 \to f^*\mathrm{Sym}^{d-2}(\wedge^2 V^*) \to \mathrm{Sym}^d(\wedge^2 V^*) \to \mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}}(d)) \to 0.$$

Dualize this sequence to get the split exact sequence

$$0 \to f\,\mathrm{Sym}^{d-2}(\wedge^2 V) \to \mathrm{Sym}^d(\wedge^2 V) \to Q_d \to 0.$$

where $Q_d = \mathrm{H}^0(\mathbb{G},\mathcal{O}\mathbb{G}(d))^*$. Since f is the trivial character it follows there is an isomorphism

$$f\,\mathrm{Sym}^d(\wedge^2 V) \cong \mathrm{Sym}^d(\wedge^2 V)$$

of $\mathrm{SL}(V)$-modules. By the Borel-Weil-Bott Theorem it follows Q_d is an irreducible $\mathrm{SL}(V)$-module. If $d = 2k$ we get by induction the equality

$$\mathrm{Sym}^d(\wedge^2 V^*) = Q_d \oplus Q_{d-2} \oplus \cdots \oplus Q_2 \oplus Q_0,$$

and the claim of the Proposition is proved in the case where $d = 2k$. The claim when $d = 2k + 1$ follows by a similar argument and the Proposition is proved.

Corollary 5.3. *Let* $\mathcal{E} = \oplus_{i=0}^l \mathcal{O}_{\mathbb{G}}(2i-d)$ *where* $l = k$ *if* $d = 2k$ *or* $d = 2k + 1$. *It follows*

$$\mathrm{H}^0(\mathbb{G},\mathcal{E}) \cong \mathrm{Sym}^d(\wedge^2 V^*)$$

as $\mathrm{SL}(V)$-module.

Proof. We get by Proposition 5.4 isomorphisms of $\mathrm{SL}(V)$-modules

$$\mathrm{H}^0(\mathbb{G},\mathcal{E}) \cong \mathrm{H}^0(\mathbb{G},\oplus_{i=0}^l \mathcal{O}_{\mathbb{G}}(d-2i)) \cong$$

$$\oplus_{i=0}^l \mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}}(d-2i)) \cong \mathrm{Sym}^d(\wedge^2 V)^* \cong \mathrm{Sym}^d(\wedge^2 V^*)$$

and the Corollary is proved.

Corollary 5.4. *There is for every* $d \geq 1$ *an equality*

$$\mathrm{Aut}_{\mathrm{SL}(V)}(\mathrm{Sym}^d(\wedge^2 V)) = \prod_{i=0}^l K^*$$

where $l = k$ *if* $d = 2k$ *or* $d = 2k + 1$.

Proof. This follows from Proposition 5.4 and the Borel-Weil-Bott theorem (BWB). From the BWB theorem it follows $\mathrm{H}^0(\mathbb{G},\mathcal{OO}\mathbb{G}(d))^*$ is an irreducible $\mathrm{SL}(V)$-module for all $d \geq 1$. From this and Proposition 5.4 the claim of the Corollary follows.

Hence the $\mathrm{SL}(V)$-module $\mathrm{Sym}^d(\wedge^2 V)$ is a multiplicity free $\mathrm{SL}(V)$-module for all $d \geq 1$. This is not true in general for $\mathrm{Sym}^d(\wedge^m K^{m+n})$ when $m, n > 2$.

In general if $\mathbb{S}_\lambda$ and $\mathbb{S}_\mu$ are two *Schur-Weyl modules* [14] there is a decomposition

$$\mathbb{S}_\lambda(\mathbb{S}_\mu(V)) \cong \oplus_i V_{\lambda_i}$$

where V_{λ_i} is an irreducible $\mathrm{SL}(V)$-module for all i. It is an open problem to calculate this decomposition for two arbitrary partitions λ and μ.

Appendix B: The Cauchy Formula

We include in this section an elementary discussion of the Cauchy formula using multilinear algebra. Let $W \subseteq V$ be vector spaces of dimension m and $m + n$ over K and let $P \subseteq \mathrm{SL}(V)$ be the subgroup fixing W. Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{p} = Lie(P)$. There is a canonical isomorphism

$$\mathfrak{g}/\mathfrak{p} \cong \mathrm{Hom}(W, V/W)$$

of P-modules, hence the elements of $\mathfrak{g}/\mathfrak{p}$ may be interpreted as linear maps. The symmetric power $\mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}) \cong \mathrm{Sym}^k(\mathrm{Hom}(W, V/W))$ is a P-module hence a $P_{semi} = \mathrm{SL}(m) \times \mathrm{SL}(n)$-module and we want to give an explicit construction of its highest weight vectors as P_{semi}-module.

Proposition 6.1. *Let* $U = K^m$. *There is a canonical map of* $\mathrm{SL}(V)$*-modules*

$$\wedge^m(U^*)\otimes\wedge^m U\to \mathrm{Sym}^m(\mathrm{Hom}(U,U))$$

defined by

$$x_1\wedge\cdots\wedge x_m\otimes e_1\wedge\cdots\wedge e_m\to\begin{vmatrix} x_1\otimes e_1 & x_1\otimes e_2 & \cdots & x_1\otimes e_m\\ x_2\otimes e_1 & x_2\otimes e_2 & \cdots & x_2\otimes e_m\\ x_m\otimes e_1 & x_m\otimes e_2 & \cdots & x_m\otimes e_m.\end{vmatrix}$$

Here e_1, .., e_m is a basis for U and x_1, .., x_m is a basis for U^*.

Proof. The proof is left to the reader as an exercise.

Note: in Proposition 6.1 the element $x_i\otimes e_j$ is an element of $U^*\otimes U$ = Hom(U,U). Hence the determinant

$$\begin{vmatrix} x_1\otimes e_1 & x_1\otimes e_2 & \cdots & x_1\otimes e_m\\ x_2\otimes e_1 & x_2\otimes e_2 & \cdots & x_2\otimes e_m\\ x_m\otimes e_1 & x_m\otimes e_2 & \cdots & x_m\otimes e_m.\end{vmatrix}$$

may be interpreted as a polynomial of degree m in the elements $x_i\otimes e_j$, hence it is an element of $\mathrm{Sym}^m(\mathrm{Hom}(U,U))$.

Let $B\subseteq SL(m,K)\times SL(n,K)\subseteq SL(V)=SL(m+n,K)$ be the following subgroup: B consists of matrices with determinant one of the form

$$\begin{pmatrix} U_1 & 0\\ 0 & U_2\end{pmatrix}$$

where

$$U_1=\begin{pmatrix} a_{11} & 0 & \cdots & 0\\ a_{21} & a_{22} & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mm}\end{pmatrix}$$

and

$$U_2\begin{pmatrix} b_{11} & 0 & \cdots & 0\\ b_{21} & b_{22} & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ b_{n1} & 0 & b_{n2}\cdots & b_{nn}\end{pmatrix}.$$

Let T be a B-module and $v\in T$ a vector with the property that for all $x\in B$ it follows

$$xv=\lambda(x)v$$

where $\lambda\in(\mathrm{Hom}(B,K^*)$ is a character of B. It follows v is a highest weight vector for T as $SL(m,K)\times SL(n,K)$-module. The group $B\subseteq SL(V)$ defines filtrations of W and V/W as follows: Let W have basis e_1, .., e_m and V have basis e_1, .., e_m, f_1, .., f_n. Let $W_1=\{e_m\}$, $W_2=\{e_m, e_{m-1}\}$, and

$W_i=\{e_m, ..e_{m-i+1}\}$.

It follows we get a filtration

$$0=W_0\subseteq W_1\subseteq\cdots\subseteq W_{m-1}=W$$

of the vector space W. Let

$$U_j=W_{m-1}\cup\{f_n,..,f_{n-j+1}\}$$

and let $V_i=(V/W)/U_{n-i}$. We get a surjection

$$V/W\to V_i$$

for $i=1,..,n-1$. It follows $dim W_i=dim V_i=d_i$ for all i. Let $x:W\to V/W$ be a linear map of vector spaces. We get an induced map

$$x_i: W_i\to V_i$$

wich is a square d_i matrix for all i. Let $g\in B$ be the element

$$\begin{pmatrix} G_1 & 0\\ 0 & G_2\end{pmatrix}$$

where

$$G_1=\begin{pmatrix} a_1 & 0 & \cdots & 0\\ & a_2 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ & * & \cdots & a_m\end{pmatrix}$$

and

$$G_2\begin{pmatrix} b_1 & 0 & \cdots & 0\\ & b_2 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ & 0 & \cdots & b_n\end{pmatrix}.$$

The i'th wedge product

$$|x_i|=\wedge^i x_i\in\mathrm{Hom}(\wedge^i W_i,\wedge^i V_i)=\wedge^i(W_i^*)\otimes\wedge^i V_i$$

may be viewed as an element in

$$|x_i|\in\mathrm{Sym}^i(\mathrm{Hom}(W_i,V_i))\subseteq\mathrm{Sym}^i(\mathrm{Hom}(W,V/W))$$

via Proposition 6.1.

Proposition 6.2. *The following formula holds:*

$$g|x_i|=\frac{b_1\cdots b_i}{a_{m-i+1}\cdots a_m}|x_i|=\lambda(g)|x_i|$$

for all $g\in B$. Here $\lambda(g)=\frac{b_1\cdots b_i}{a_{m-i+1}\cdots a_m}$ is a character $\lambda\in\mathrm{Hom}(B,K^*)$.

Proof. The proof is left to the reader as an exercise.

Hence the i'th determinant $|x_i|\in\mathrm{Sym}^i(\mathrm{Hom}(W,V/W))$ is a highest weight vector for the $SL(m)\times SL(n)$-module $\mathrm{Sym}^i(\mathrm{Hom}(W,V/W))$. By the results of studies Brion [17-22], it follows the vectors $x_0^{d_0}x_1^{d_1}\cdots x_i^{d_i}$ with $\sum id_i=k$ are all highest weight vectors for the module

$\mathrm{Sym}^k(\mathrm{Hom}(W,V/W))\cong\mathrm{Sym}^k(W^*\otimes V/W)$.

Acknowledgements

The author thanks Michel Brion, Alexei Roudakov and an anonymous referee for comments on the contents of this paper.

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Citation: Maakestad H (2015) Jet Bundles on Projective Space II. J Generalized Lie Theory Appl S2: 001. doi:10.4172/1736-4337.S2-001

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, Unversity of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications

Randriambololondrantomalala, J Generalized Lie Theory Appl 2015, S2
http://dx.doi.org/10.4172/1736-4337.S2-002

Research Article Open Access

The *m*-Derivations of Analytic Vector Fields Lie Algebras

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Abstract

We consider a (real or complex) analytic manifold *M*. Assuming that *F* is a ring of all analytic functions, full or truncated with respect to the local coordinates on *M*; we study the (m ≥ 2)-derivations of all involutive analytic distributions over *F* and their respective normalizers.

Keywords: *m*-derivations; Analytic vector fields Lie algebras; Distributions; Generalized foliations; Stein manifolds; Compact holomorphic manifolds; Chevalley-Eilenberg's cohomology; Compactly supported vector fields

Introduction and Preliminary

We know several embedding theorems in differential geometry, some of them are of John F. Nash in Riemannian manifolds [1,2], of Whitney [3] in differentiable manifolds and of Grauert in analytic manifolds cf. [4]. They make easy certain study on a differentiable manifold. Here, we are interested to a real or complex analytic *n*-dimensional manifold *M* and let *F*(*M*) be the ring of all analytic functions on *M*. We know that these manifolds can be considered as smooth manifolds. But certain property on a smooth manifold cannot be true on *M*, for example the global representation of a smooth function germ theorem. Grabowski had this problem when he studied derivations of the real or complex analytic vector fields Lie algebra cf. [5] and he used Stein manifolds to avoid technical difficulties in them. Here, we examine not only the derivations but the ($m \geq 2$)-derivations (generalization of derivation's notion) of a Lie subalgebra of the real or complex analytic vector fields Lie algebra on *M*, using Lie algebra tools. In advance, we state that the considered Lie algebras have enough sections more than constant ones in the Lie algebra of all analytic vector fields. Then, we consider only Stein spaces unless expressly stated in a complex analytic case. In the real analytic one, we don't need more hypothesis because of the imbedding theorem of Grauert and Cartan theorems [6]. More precisely, any real analytic manifold can be considered as a closed submanifold of a certain $\mathbb{R}^l$ (a " real Stein manifold"). Now, an m-derivation of a Lie algebra *A* is a linear map from *A* to itself which is distributive on the brackets $[X_1,[X_2,\ldots[X_{m-1},X_m]\ldots]]$, where all X_i are in *A*. On the one hand, we have studied the *m*-derivations of polynomial vector fields Lie algebras on $\mathbb{R}^n$ in studies of 7. Randriambololondrantomalala [7], an important Lie subalgebra of analytic vector fields, we found that Lie algebras of derivations are different to those of ($m > 2$)-derivations. One can see the following example, on $\mathbb{R}^3$, the Lie $\mathbb{R}$-algebra is spanned by $\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},y\frac{\partial}{\partial x},(y)^2\frac{\partial}{\partial x}$ and let's define the $\mathbb{R}$–linear map *D* by $D\left((y)^2\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial x}$ which is zero otherwise. It's clear that *D* is not a derivation, but a 3-derivation. On the other hand, all *m*-derivations of a distribution over the full or truncated rings of smooth functions on a differentiable manifold in literature of Randriambololondrantomalala [8], are derivations. These facts lead us to ask if a distribution Lie algebra on an analytic manifold has results as the one or the other above results. So, we will divide our paper into three parts. First, we take a real or complex analytic involutive distribution Ω over *M*. That is to say, Ω is a *F*(*M*)-submodule of the analytic vector fields Lie algebra $\chi(M)$ on *M*. We can find some examples of these distributions and the interests for studying their derivations in literature of Grabowski and Cartan [5,6]. Here, we find the Ω's centralizer and the derivative ideal of Ω. We can say also that the normalizer of Ω is a Lie subalgebra of analytic vector fields. In addition, we find out that all *m*-derivations of Ω (resp. of the normalizer of Ω) are inner with respect to a normalizer's vector field (resp. are inner). Second, assuming that Ω is an involutive distribution on *M* over a subring *F* of *F*(*M*), namely an *F*-submodule of $\chi(M)$ stable by the vector fields bracket, where *F*≠ *F*(*M*). One can consider a system of commuting vector fields on *M* as in studies of Randriambololondrantomalala [8] and all distribution Lie subalgebras of the Lie algebra of analytic vector fields which commute with this system. The normalizer of Ω is an analytic vector fields Lie algebra and contains locally all constant vector fields and Euler's vector field. But in general, we can't use the reasoning by Randriambololondrantomalala [7] to characterize *m*-derivations of Ω. We make more explicit all *m*-derivations of Ω and of some of its normalizer. Whereas, in the end, we discuss the Lie algebras of holomorphic vector fields, especially when the holomorphic manifold is not a Stein one, and Lie algebras of locally polynomial vector fields on an analytic manifold *M*. Their *m*-derivations as well as their normalizers can be characterized by using some results of Randriambololondrantomalala [7].

Therefore, we have found the *m*-derivations of all distributions over a set of full or truncated analytic functions with respect to the local coordinates on *M*. When $m = 2$, we deduce from our results some [5]'s theorems. Third, we can apply our theorems on Lie algebras of real or complex analytic vector fields on *M*, of generalized foliation on *M* cf. [9], a Lie subalgebra of analytic vector fields on $\mathbb{S}^2$ and on $\mathbb{T}^2$, Riemann surfaces, etc. Relations between the Lie algebra of compactly supported vector fields and the compactness of *M* are discussed. Moreover, we emphasize the extensions of our theorems when the studied distributions are singular, by using the complexification of a real analytic manifold, Hartogs and Riemann extension theorems. Of course, in these circumstances, we can use theory of coherent sheaves

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Received November 23, 2015; **Accepted** December 17, 2015; **Published** December 19, 2015

Citation: Randriambololondrantomalala P (2015) The *m*-Derivations of Analytic Vector Fields Lie Algebras. J Generalized Lie Theory Appl S2: 002. doi:10.4172/1736-4337.S2-002

made by Cartan [6] in a Stein case or pass into Grabowski's conjecture cf. [9]. We interpret our results in Chevalley-Eilenberg cohomology sense when $m = 2$.

Following the above notations, let M be a real or complex analytic n-dimensional manifold. In complex case, we regard a Stein manifold unless special mention. We denote by $\chi(M)$ the Lie algebra of analytic vector fields on M and $F(M)$ the ring of analytic functions on M. Throughout this paper, we take an atlas in which every chart are connected. Then, the open subset of a chart U where a non-trivial subset of $\chi(M)$ doesn't vanish, is dense on U (non-trivial means different to {0}). We can use certain results of Randriambololondrantomalala [7,8] because in the proofs of theorem of these papers we consider only analytic functions (polynomials, exponentials). In the same way, we don't need partition of the unity to make global some local results cf. [10]. In all sections of this article, we set an integer $m \geq 2$, recall that D is an m-derivation of a Lie algebra A if for $(X_i)_{1\leq i\leq m} \subset A$, we get $D[X_1,[X_2,\ldots,[X_{m-1},X_m]\ldots]] = [D(X_1),[X_2,\ldots,[X_{m-1},X_m]\ldots]] + \ldots + [X_1,[X_2,\ldots,[X_{m-1},D(X_m)]\ldots]]$. This D is said inner on a Lie algebra $\mathbb{B}$ containing A, if D is a Lie derivative with respect to an element of $\mathbb{B}$. Recall us another basic definition cf. [11].

Definition 1.1. A complex manifold M is a Stein manifold, if we have simultaneously the three following conditions: For every $x \neq y$, both in M; there is a holomorphic function f over M such that $f(x) \neq f(y)$. For all $x \in M$, it exists n holomorphic functions (f_i) over M such that df_i are linearly independent over $\mathbb{C}$ on x. If K is a compact set of M, the following set is compact (holomorphic convexity of M)

$$\left\{x \in M \,/\, |f(x)| \leq \sup_{y\in K} |f(y)|, \text{for all holomorphic functions } f \text{ over } M\right\}.$$

From these assertions, every local ring of holomorphic functions around $x \in M$ is spanned by holomorphic functions on M cf. [12].

Some results of the Lie algebra of compactly supported vector fields C_c relative to a Stein manifold are the following,

Proposition 1.2. *A complex analytic manifold M is compact iff C_c is non trivial, particularly if M is a Stein holomorphic manifold, C_c is trivial.*

Proof. It's obvious that if M is compact, then $\mathfrak{C}_c = \chi(M)$ and C_c is not trivial. Conversely, suppose that M isn't a compact set and there is $X \in C_c$ such that $K = \mathrm{Supp}(X) \neq \emptyset$. We can consider $K \neq M$ because M is not compact. Then, we have the nullity of X in the open set $\complement K \neq \emptyset$. By analyticity, X vanishes in whole M. Hence, we have a contradiction about $K \neq \emptyset$ and we obtain M is a compact set. It's clear that a Stein space is never a compact set by definition, then its Lie algebra of compactly supported vector fields is trivial.

The *m*-derivations defined by distributions on *F*(*M*)

Let Ω be a non-trivial involutive analytic distribution over the analytic functions ring on M. Let N be the normalizer of Ω in $\chi(M)$, that is to say that the set of all $X \in \chi(M)$ such that $[X,\Omega] \subset \Omega$, and $\mathfrak{B} = \{x \in M \,/\, \Omega(x) \neq \{0\}\}$. We can choose a connected domain U_i of a chart. If we suppose that it exists an open set O_i in U_i where Ω vanishes, then $\Omega_{|U_i} = \{0\}$ by analyticity. Otherwise, every open set in U_i contains an element of **B**. So, $\mathbf{B} \cap U_i$ is dense over U_i. Moreover, the collection of U_i forms an atlas of M, then **B** is dense over M. The set **B** is an open analytic submanifold of M. Particularly, **B** is a Stein cf. [13]. Thus, every vector field defined over **B** admits a continuous extension on M, and if this last one is analytic, then it's necessarily an element of the normalizer of Ω. We use this last fact when we deal with extension theorems.

We know by literature of Nagano's [14] result that Ω is integrable, then it yields a generalized foliation F on M cf. [10]. So, Ω is the Lie algebra of tangent vector fields to the foliation and L_F the one of all foliation preserving vector fields. It is known that the normalizer N in $\chi(M)$ of Ω contains L_F cf. [10]. Hence, the restriction of the foliation in **B** is non singular.

Proposition 2.1. *The centralizer of Ω vanishes and the derivative ideal of Ω coincides with Ω itself.*

Proof. We say that $X \in \chi(M)$ is in the centralizer of Ω if $[X, \Omega]=\{0\}$; and the derivative ideal of denoted by $[\Omega, \Omega]$ is the Lie algebra spanned by all brackets of two elements of Ω. Suppose there is an non zero element X of the centralizer, we have $[X, f\Omega] = (X(f))\Omega = \{0\}$, for all $f \in F(M)$. It's not possible in a Stein manifold or in a real analytic manifold if X doesn't vanish identically over M and if $\Omega \neq \{0\}$. Along with this result, we can adapt the proof of Proposition 2.28 of studies of Randriambololondrantomalala [15] and assert that $[\Omega, \Omega] = \Omega$.

Let's recall an Hartogs's extension theorem and Riemann extension theorem.

Theorem 2.2. *(Hartogs [16]) Let be $t \geq 2$ and D be a bounded domain in $\mathbb{C}^t$. In addition, K be a compact subset of D such that $D - K$ is a connected set. Then all holomorphic functions f over $D - K$ can be extended holomorphically to D.*

Theorem 2.3. *(Riemann extension theorem) Let U be an open set in $\mathbb{C}$ and $z_0 \in U$. If $f: U - \{z_0\} \to \mathbb{C}$ is holomorphic function such that z_0 is a removable singularity of f, then f can be extended into an unique holomorphic function $\bar{f}$ in U where $\bar{f}(z_0) = \lim_{z\to z_0} f(z)$.*

Theorem 2.4. *In holomorphic case, all m-derivations of Ω, L_F, and of* N *are Lie derivatives with respect to elements belonging to* N. *In real analytic one, we have the same results if* **B** $= M$.

Proof. We can prove this assertion over **B** by Theorem 2.1 of studies of Randriambololondrantomalala [8] using Proposition 2.1 and partially Theorem 3.7 of literature of Randriambololondrantomalala [10]. For the corresponding extension theorem over M, we adopt the following arguments. We know that **B** is dense over M, then the restriction of **B** in each domain of a chart U is dense over U (U is a bounded set). The complement of this $\mathbf{B} \cap U$ in U can be considered as a compact set of the chart such that $\mathbf{B} \cap U$ is connected. In holomorphic case, when $n \geq 2$, we use Hartogs's theorem in a domain of the chart, so the extension theorem over M holds. If $n = 1$, we know by the isolated zeros principle that the domain of chart contains only a finite number of zeros in the corresponding restriction of **B**. By continuity at these zeros, which are removable singularities, the Riemann extension theorem can be used. Of course, if **B** $= M$ in real analytic situation, the extension theorem is applicable.

Definition 2.5. The complexification of a real analytic manifold M is a holomorphic manifold $\mathbb{H}$ such that there is a real analytic embedding $f: M \to \mathbb{H}$ where $\mathbb{H}$ has a holomorphic atlas $(U_i, \varphi_i)_i$ and $\varphi_i(f(M) \cap U_i) = \varphi_i(U_i) \cap \mathbb{R}^n$. We have a Stein complexification if $\mathbb{H}$ is Stein.

The next theorem is due by Grauert cf. [4,12].

Theorem 2.6. *Every real analytic manifold has a Stein complexification and can be analytically properly embedded into an Euclidean space $\mathbb{R}^N$.*

The following complexification of a Lie subalgebra $\mathbb{G}$ of the real analytic vector fields Lie algebra of M is in the following sense: if M can be embedded in a holomorphic manifold $\mathbb{H}$, the complexification $\overline{\mathbb{G}}$ of $\mathbb{G}$ in $\mathbb{H}$ is such that $\overline{\mathbb{G}}_{|M} = \mathbb{G}$.

Theorem 2.7. *If the complexification of Ω in a Stein space T is still an involutive distribution, then the first result of the Theorem 2.4 holds in real analytic case.*

Proof. We use the complexification of M on a Stein space T. Consequently, let be $\overline{\Omega}$ the complexification over T of Ω. Recall that $\overline{\Omega}$ is an involutive distribution over T where its normalizer on $\chi(T)$ is denoted by N_0. So, all m-derivations of Ω, of L_F, and of N have their complexified m-derivations over T on respectively $\overline{\Omega}$, the Lie algebra of all foliation preserving vector fields $\overline{L_{\mathfrak{F}}}$ on T and N_0. By the results of Theorem 2.4, these last m-derivations are Lie derivatives with respect to elements belonging to N_0. We can affirm that $\mathfrak{N}_{0|M} = \mathfrak{N}$ and $\overline{L_{\mathfrak{F}}}_{|M} = L_{\mathfrak{F}}$ by $\overline{\Omega}_{|M} = \Omega$. Thus, we have the same result as in the first part of Theorem 2.4.

By definition, the first space of Chevalley-Eilenberg's cohomology of a Lie algebra $\mathcal{A}$ denoted by $H^1(\mathcal{A})$ is $\mathrm{Der}(\mathcal{A})/\mathrm{ad}_{\mathcal{A}}$ with $\mathrm{Der}(\mathcal{A})$ the Lie algebra of all derivations of $\mathcal{A}$ and $\mathrm{ad}_{\mathcal{A}}$ the set of inner derivations of $\mathcal{A}$.

Throughout this paper, we suppose that all hypothesis of Theorem 2.7 are satisfied or $\mathfrak{B} = M$, in real analytic case.

Following ideas of Theorem 3.7, Corollary 2.14 and Remark 2.15 of literature of Randriambololondrantomalala [10], we state

Corollary 2.8. *The first space of Chevalley-Eilenberg's cohomology of Ω, $L_{\mathfrak{F}}$, and of $\mathfrak{N}$ is respectively isomorphic to the following respective Lie algebras, $\mathfrak{N}/\Omega$, $\mathfrak{N}/L_{\mathfrak{F}}$, $\{0\}$.*

Remark 2.9. *By Theorem 2.4, we deduce Theorem 3.2 and 4.1 of studies of Grabowski* [5].

The *m*-derivations associated to a distribution over a subring of *F*(*M*)

Let be an atlas of M such that Ω is locally spanned by $\left(\frac{\partial}{\partial x^i}\right)_{1\le i\le n}$ over the ring F_0 of real or complex functions depending only on $(x^j)_{1\le j\le k}$ with respect to the atlas (where $1 \le k < n$). We can consider Ω to be a Lie algebra which commutes with a system S of commuting vector fields by the usual bracket. That is to say, $S = \{X_1,\ldots,X_q\}$ such that $[X_i, X_j] = 0$ and S is locally of rank $n - k$ ($0 < q \le n$). It is easy to check that $[\Omega,\Omega] = \Omega$ because of Randriambololondrantomalala's [8] result. So with the same reason, every m-derivation of Ω is local. Moreover, the normalizer N of Ω is locally isomorphic to $\Omega \oplus gl(n-k, \mathbb{R} \text{ or } \mathbb{C})$ as a vector space. We consider the closed 1–differential forms α^i and ω^i over a $(n - k)$-dimensional distinguished connected chart of the generalized foliation generated by S, where $i = k+1,\ldots,n$ and an m-derivation of Ω, $D^{(\alpha,\omega)} = \sum_{k+1\le j\le n}(\alpha^j + \omega^j)\otimes\frac{\partial}{\partial x^j}$ such that $ker(\omega^j) \supset F_0(U)\left\langle\frac{\partial}{\partial x^i}\right\rangle_{1\le i\le k}$ and $ker(\alpha^j) \supset F_0(U)\left\langle\frac{\partial}{\partial x^i}\right\rangle_{k+1\le i\le n}$ (S in this chart is $\{\frac{\partial}{\partial x^j}\}_{k+1\le j\le n}$, $F\langle A\rangle$ denotes a module spanned by A over a ring F) cf. [8]. We have omitted all singular charts of the foliation because the open set R of all regular points is dense over M cf. [10], we have no problem for the extension of our results from R towards M as in the previous section. By adapting Theorem 3.12 of literature of Randriambololondrantomalala [8], we obtain easily

Theorem 3.1. *All m-derivations of Ω (resp. of $\mathfrak{N}$) are a sum of a Lie derivative with respect to one element of $\mathfrak{N}$ and a derivation $D^{(\alpha,w)}$ as denoted before (resp. are similar to m-derivations of Ω).*

Hence, adopting the arguments of Theorem 3.19 of studies of Ravelonirina [17], we hold the following

Corollary 3.2. *When the rank of S is a positive constant $n - k$, the first spaces of Chevalley-Eilenberg's cohomology of Ω and of N are both isomorphic to $(H_R(B))^{n-k} \times \left(\mathbb{R}^{(n-k)^2+n-k} \text{ or } \mathbb{C}^{(n-k)^2+n-k}\right)$ with $H_R(B)$ is the de Rham cohomology of foliation basic forms of M.*

As we know, we can split the above Ω into a semi-direct sum of Lie algebras Ω_S^1 and Ω_S^2 as in studies of Randriambololondrantomalala [8], where they are modules on the ring $F_0(M)$ of constant functions over the leaves relative to the above generalized foliation. We notice that Ω_S^2 is spanned by S on $F_0(M)$. We can reason on a distinguished chart U with the coordinates $(x^i)_{1\le i\le n}$. The $F_0(U)$ is the set of all analytic functions depending only on $(x^i)_{1\le i\le k}$, $\Omega^1_{S|U}$ is spanned by $\left(\frac{\partial}{\partial x^i}\right)_{1\le i\le k}$ and $\Omega^2_{S|U}$ by $\left(\frac{\partial}{\partial x^i}\right)_{k+1\le i\le n}$ on $F_0(U)$.

Now, we discuss the m-derivations of Ω_S^1. The normalizer N^1 of this Lie algebra can be written as a direct sum of Lie algebras $\mathfrak{N}^1 = [\Omega_S^1, \Omega_S^1] \oplus [\mathcal{Q},\mathcal{Q}]$, where $\mathcal{Q}$ is the centralizer of Ω_S^1 in $\chi(M)$ and the center of Ω_S^1 is zero ($\mathcal{Q}$ is locally spanned by $\left(\frac{\partial}{\partial x^i}\right)_{k+1\le i\le n}$ on the ring of all analytic functions depending only on $(x^i)_{k+1\le i\le n}$). By a similar argument of Nakanishi [18], we deduce that all m-derivations of N^1 are a direct sum of those of Ω_S^1 and of $\mathcal{Q}$. By Theorem 2.4, it's clear that

Theorem 3.3. *Each m-derivation of Ω_S^1 (resp. of $\mathfrak{N}^1$) is a Lie derivative with respect to an element of* N^1.

The normalizer of Ω_S^2 is locally the sum of the $F_0(U)$ -module spanned by $\underset{1\le i\le n}{\frac{\partial}{\partial x^i}}$ and a vector space generated by $\underset{k+1\le i,\, l\le n}{x^l\frac{\partial}{\partial x^i}}$. That is to say, its normalizer is $\mathfrak{N}$. In addition, its centralizer is Ω_S^2 itself. Because of $[\Omega_S^2, \Omega_S^2] = \{0\}$ or Ω_S^2 is nilpotent of order 1, we obtain easily

Theorem 3.4. *Every endomorphism of Ω_S^2 is an m-derivation of Ω_S^2.*

So, it's immediate that

Corollary 3.5. *The first space of Chevalley-Eilenberg's cohomology of Ω_S^1, Ω_S^2 and of N^1 are respectively isomorphic to the following respective Lie algebras, $\mathfrak{N}^1/\Omega_S^1$, $\mathrm{End}(\Omega_S^2)/\Omega_S^2$, $\{0\}$.*

Let's consider Ω_S^3 the Lie subalgebra of Ω, spanned by X_i over a ring $F \subseteq F(M)$. That is to say, F is locally the set of all analytic functions depending only on $(x^i)_{l\le i\le n}$ where $0 < l < k + 1$ (resp. $k + 1 < l < n + 1$). When $F = F(M)$ (resp. $F = F_0(M)$), it is a special case of Lie algebras defined in Theorem 2.4 (resp. in Theorem 3.1) when the submodule is generated by X_i. In the distinguished local coordinates, Ω_S^3 is spanned by $\left(\frac{\partial}{\partial x^j}\right)_{k+1\le j\le n}$ over F. The normalizer $\mathfrak{N}^3$ of Ω_S^3 coincides with the sum of Ω_S^3 and Ω_S^4 where the element of this last one is locally the following analytic vector fields $\sum_{l\le i\le k} f^i(x^t, l\le t\le k)\frac{\partial}{\partial x^i} + \sum_{1\le j\le l-1} g^j(x^t, 1\le t\le k)\frac{\partial}{\partial x^j}$

$\left(\text{resp.}\, f^i(x^t, 1 \le t \le k)\underset{1\le i\le k}{\frac{\partial}{\partial x^i}} + h^j(x^t, l \le t \le n)x^j \underset{k+1\le j\le l-1}{\frac{\partial}{\partial x^j}}\right)$. In the first case, we can adapt Theorem 2.4 because all analytic functions depending on (x^j) where $k+1 \le j \le n$, are in the base ring of Ω_S^3. In the following case, it is easy to see that Theorem 3.1 can be adapted to Ω_S^3. Thus

Theorem 3.6. *In the first case, every m-derivation D of Ω_S^3 is a Lie derivative with respect to a $\mathfrak{N}^3$'s element; in the second, it is a sum of a Lie derivative of an element of $\mathfrak{N}^3$ and a $D^{(\alpha,w)}$ analogous to that of Theorem 3.1. In addition, the corresponding extension theorems hold.*

Corollary 3.7. *The first space of Chevalley-Eilenberg's cohomology of Ω_S^3 is respectively isomorphic to Ω_S^4 in the first case;*

$$\Omega_S^4 \oplus \left(\left(\mathrm{H}_R(\mathrm{B}')\right)^{l-k-1} \times \left(\mathbb{R}^{(l-k)(l-k-1)} \text{ ou } \mathbb{C}^{(l-k)(l-k-1)}\right)\right)$$

in the other one if *S* has a constant rank ($\oplus$ is a module direct sum and *B′* is the set of the corresponding foliation basic forms of *M*).

When we regard all the above normalizers on a distinguished chart, they contain locally all constant fields and Euler vector field. So, we ask one question: could we adapt Theorem 3.6 and Theorem 3.9 in [7] to these normalizers? The following remark shows us that this argument is false.

Remark 3.8. On $\mathbb{R}^3$, we consider the Lie $\mathbb{R}$-algebra *A* spanned by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y\frac{\partial}{\partial x}, (y)^2\frac{\partial}{\partial x}, e^y\frac{\partial}{\partial x}, ye^y\frac{\partial}{\partial x}, y^2e^y\frac{\partial}{\partial x}, \ldots$. Lemma 2.3 of literature of Randriambololondrantomalala [10] is not true for *A*, so the arguments proposed in the proof of Theorem 3.6 of Princy [7] don't hold in this situation.

Whereas, let P be a Lie subalgebra consisting locally of polynomial vector fields in $\chi(M)$, where the Euler vector field and all constant vector fields are locally in P. Especially, *M* is not supposed to be a Stein in the holomorphic case. Let's recall a well known theorem,

Theorem 3.9. *(The maximum principle)* [12] *Let be M a connected holomorphic manifold and f a holomorphic function on M such that $|f(z)| \le |f(z_0)|$, where $z_0 \in M$ for all $z \in M$; then f is a constant function.*

One consequence of the maximum principle is the following, if the holomorphic manifold *M* is compact, every holomorphic function on *M* is constant in every connected component of *M*. We know that *M* is locally connected, then each function over *M* is locally constant. Therefore, it's clear that if *M* is a compact and connected holomorphic manifold, the ring of all holomorphic functions on *M* is the complex constant functions ring. It's confirm that results of the following theorem complete our study about an involutive analytic distribution when *F*(*M*) is reduced to $\mathbb{C}$.

By adapting Randriambololondrantomalala's [7] theorems and taking account that the vector field found in the proof of Theorem 3.6 of Princy [7] is analytic, it follows that

Theorem 3.10. *When m is even, all m-derivations of $\mathfrak{P}$ (resp. of the normalizer $\mathcal{N}$ of $\mathfrak{P}$ in $\chi(M)$) are a Lie derivative with respect to one and only one vector field belonging to $\mathcal{N}$ (resp. to the normalizer of $\mathcal{N}$ in $\chi(M)$). If m is an odd number, they are sum of a Lie derivative with respect to one element of $\mathcal{N}$ (resp. of the normalizer of $\mathcal{N}$) and an m-derivation of local homogeneous degree -2 of $\mathfrak{P}$.*

So, taking into account: the vanishing of the centralizer of $\mathfrak{P}$ cf. [19] p.91; both the proofs of Theorem 2.12 of Ravelonirina [19], Corollary 3.12 of Randriambololondrantomalala [7] and Theorem 3.7 in literature of Randriambololondrantomalala [10], we obtain

Corollary 3.11. *The first space of Chevalley-Eilenberg's cohomology of $\mathfrak{P}$, of $\mathcal{N}$ and of* N *is respectively isomorphic to the following respective Lie algebras $\mathcal{N}/\mathfrak{P}$,* N / $\mathcal{N}$, {0}, *where* N *is the normalizer of $\mathcal{N}$.*

Illustrations

Some illustrations of our theorems are given in this section.

Example 4.1. It's clear that Theorem 2.4 works well on the Lie algebra of all analytic vector fields $\chi(M)$, that is to say, all *m*-derivations of $\chi(M)$ are Lie derivatives by elements of $\chi(M)$. We can define the Lie algebra of compactly supported real analytic vector fields C and this theorem holds for this last one. In particular, $\mathrm{H}^1(\mathfrak{C}) = \chi(M)/\mathfrak{C}$ for a non-trivial $\mathfrak{C}$. More, $\mathrm{H}^1(\mathfrak{C}) = \{0\}$ if and only if *M* is compact. Obviously, we can use the above cited theorem on the Lie algebras of vector fields relative to a generalized foliation over M. We can cite some well known Stein spaces, $\mathbb{C}^n$, an open poly-disc in $\mathbb{C}^n$, non-compact Riemann surfaces, ... and build our results in these.

Example 4.2. Let be $\mathbb{S}^2$ a holomorphic compact connected manifold. It's not a Stein manifold nor a submanifold of $\mathbb{C}^u$ for any *u*, it's a compact Riemann surface. We choose the modified stereographic coordinates over this manifold. The $\mathbb{S}^2$ has an atlas composed by two charts (U, z_1) and (V, z_2) with the overlap map $\psi(z) = z^{-1}$ in $U \cap V$. We remark that the Lie algebra $\mathbb{L}$ of vector fields on *M* spanned over $\mathbb{C}$ by Y_1, Y_2 and Y_3 is the one of all polynomial vector fields in $\mathbb{S}^2$, where

$$Y_1 : \begin{cases} \frac{\partial}{\partial z^1} & \text{in } U \\ -(z^2)^2\frac{\partial}{\partial z^2} & \text{in } V \end{cases}, \qquad Y_2 : \begin{cases} -(z_1)^2\frac{\partial}{\partial z^1} & \text{in } U \\ \frac{\partial}{\partial z^2} & \text{in } V \end{cases},$$

$$Y_3 : \begin{cases} -z^1\frac{\partial}{\partial z^1} & \text{in } U \\ z^2\frac{\partial}{\partial z^2} & \text{in } V \end{cases}.$$

By Theorem 3.10, all *m*-derivations of $\mathbb{L}$ are Lie derivatives with respect to a vector field in $\mathbb{L}$ itself. That is to say, if *D* is an *m*-derivation of $\mathbb{L}$ defined by $D(Y_1) = \alpha^i Y_i$, $D(Y_2) = \beta^i Y_i$, $D(Y_2) = \gamma^i Y_i$; we have $D = \mathrm{L}_{-\gamma^1 Y_1 + \gamma^2 Y_2 + \alpha^1 Y_3}$.

When we look at $\mathbb{S}^2$ as a real analytic manifold, we set the charts $(U,(x^1,x^2))$ and $(V,(y^1,y^2))$ with the overlap map

$$\phi(x^1,x^2) = \left(y^1 = \frac{x^1}{(x^1)^2 + (x^2)^2}, y^2 = \frac{x^2}{(x^1)^2+(x^2)^2}\right).$$

We set the real analytic vector field $Y_3 : \begin{cases} -x^1\frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^2} & \text{in } U \\ y^1\frac{\partial}{\partial y^1} + y^2\frac{\partial}{\partial y^2} & \text{in } V \end{cases}$ on $\mathbb{S}^2$

and the Lie algebra A of real analytic vector fields which commute with Y_3. This $\mathfrak{A}$ consists of real analytic vector fields *Y* such that

$$Y : \begin{cases} F_1\left(\frac{x^2}{x^1}\right)x^1\frac{\partial}{\partial x^1} + F_2\left(\frac{x^1}{x^2}\right)x^2\frac{\partial}{\partial x^2} & \text{in } U \\ F_3\left(\frac{y^2}{y^1}\right)y^1\frac{\partial}{\partial y^1} + F_4\left(\frac{y^1}{y^2}\right)y^2\frac{\partial}{\partial y^2} & \text{in } V \end{cases}$$

where in the $U \cap V$,

$$F_3\left(\frac{y^2}{y^1}\right) = \frac{1}{(y^1)^2+(y^2)^2}\left(\left(-2F_2\left(\frac{y^1}{y^2}\right) + F_1\left(\frac{y^2}{y^1}\right)\right)(y^2)^2 - F_1\left(\frac{y^2}{y^1}\right)(y^1)^2\right)$$ and

$$F_4\left(\frac{y^1}{y^2}\right)=\frac{1}{(y^1)^2+(y^2)^2}\left(\left(-2F_1\left(\frac{y^2}{y^1}\right)+F_2\left(\frac{y^1}{y^2}\right)\right)(y^1)^2-F_2\left(\frac{y^1}{y^2}\right)(y^2)^2\right)$$

with F_k are arbitrary convenient functions of one variable. So, we can apply all theorems in Section 3 to $\mathfrak{A}$. Particularly $\mathrm{H}^1(\mathbb{L})=\{0\}$ and $\mathrm{H}^1(\mathfrak{A})=H^1_R(B)\times\mathbb{R}^2$.

Example 4.3. Indeed, Theorem 3.10 can be applied to a polynomial vector fields Lie algebra on the real analytic manifold $\mathbb{R}^n$ or the Stein manifold $\mathbb{C}^n$ having the corresponding hypothesis.

Example 4.4. We set the Lie algebra $\mathbb{A}$ over the Stein manifold $\mathbb{C}^3$ spanned over $\mathbb{C}$ by $\frac{\partial}{\partial z^1},\frac{\partial}{\partial z^2},\frac{\partial}{\partial z^3},z^1\frac{\partial}{\partial z^1}+z^2\frac{\partial}{\partial z^2}+z^3\frac{\partial}{\partial z^3},z^2\frac{\partial}{\partial z^1},(z^2)^2\frac{\partial}{\partial z^1}$. The normalizer of $\mathbb{A}$ is $\mathcal{N}_0=\mathbb{A}+R$, where R is the space over $\mathbb{C}$ generated by $z^1\frac{\partial}{\partial z^1},z^2\frac{\partial}{\partial z^2},z^3\frac{\partial}{\partial z^3},z^2\frac{\partial}{\partial z^3},z^3\frac{\partial}{\partial z^1}$. It is permit to use Theorem 3.10 and when m is even, every m-derivation of $\mathbb{A}$ is inner on $\mathcal{N}_0$. If m is odd, the m-derivation is a sum of an inner derivation on $\mathcal{N}_0$ with a $\mathbb{C}$-linear map D defined by $D\left((z^2)^2\frac{\partial}{\partial z^1}\right)=\alpha\frac{\partial}{\partial z^1}$ which is zero otherwise ($\alpha\in\mathbb{C}$). Moreover, all m-derivations of $\mathcal{N}_0$ are inner for all $m\geq 2$. So, $\mathrm{H}^1(\mathbb{A})\cong R\ominus\left\langle z^1\frac{\partial}{\partial z^1}+z^2\frac{\partial}{\partial z^2}+z^3\frac{\partial}{\partial z^3}\right\rangle$ and $\mathrm{H}^1(\mathcal{N}_0)\cong\{0\}$.

Remark 4.5. In the following example, Theorem 3.10 cannot be applied. We take the 2-torus $\mathbb{T}^2=\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$, which is a holomorphic connected compact manifold cf. [20], it's not a Stein. All overlap maps are translations, that is to say, they are holomorphic. We can define globally the Lie algebra of all constant vector fields $\mathfrak{Q}$ on M and find that $\mathfrak{Q}$ is the Lie algebra of all holomorphic vector fields over M. All endomorphisms of each Lie subalgebra of $\mathfrak{Q}$, which is inevitably nilpotent of order 1, are m-derivations of this subalgebra. The normalizer of this subalgebra or its centralizer is the Lie algebra of all vector fields over $\mathbb{T}^2$. But $\mathrm{H}^1(\mathfrak{Q})\cong\mathrm{End}(\mathfrak{Q})/\mathrm{ad}_{\mathfrak{Q}}$ and $\mathrm{H}^1(\chi(M))\cong\mathrm{End}(\chi(M))/\mathrm{ad}_{\chi(M)}$ since $\mathrm{H}^1(\chi(M))=\{0\}$ in smooth cases.

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Citation: Randriambololondrantomalala P (2015) The *m*-Derivations of Analytic Vector Fields Lie Algebras. J Generalized Lie Theory Appl S2: 002. doi:10.4172/1736-4337.S2-002

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, University of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications

Onsory and Araskhan, J Generalized Lie Theory Appl 2015, S2
http://dx.doi.org/10.4172/1736-4337.S2-003

Research Article Open Access

The Generalization of the Stalling's Theorem

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Abstract

In this paper, we present a relative version of the concept of lower marginal series and give some isomorphisms among $\mathcal{V}$G-marginal factor groups. Also, we conclude a generalized version of the Stalling's theorem. Finally, we present a sufficient condition under which the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of its factor groups.

Keywords: Schur-Baer variety; Pair of groups; $\mathcal{V}$G-marginal series

Introduction

There exists a long history of interaction between Schur multipliers and other mathematical concepts. This basic notion started by Schur [1], when he introduced multipliers in order to study projective representations of groups. It was known later that the Schur multiplier had a relation with homology and cohomology of groups. In fact, if G is a finite group, then $M(G)\cong H^2(G,\mathbb{C}^*)\cong H_2(G,\mathbb{Z})$, where $M(G)$ is the Schur multiplier of G, $H^2(G, \mathbb{C}^*)$ is the second cohomology of G with coefficient in $\mathbb{C}^*$ and $H_2(G, \mathbb{Z})$ is the second internal homology of G [2]. Hopf [3] proved that $M(G)\cong (R\cap F^2)/[R,F]$. He also proved that the Schur multiplier of G is independent of the free presentation of G. Let (G, N) be a pair of groups, where N is a normal subgroup in Ellis [4] defined the Schur multiplier of the pair (G, N) to be the abelian group $M(G, N)$ appears in the following natural exact sequence

$$H_3(G)\to H_3(G,N)\to M(G,N)\to M(G)\to M(G/N)$$
$$\to G/[N,G]\to (G)_{ab}\to (G/N)_{ab}\to 1,$$

where $H_3(-)$ denote the third homology of a group with integer coefficients. He also proved that if the normal subgroup N possess a complement in G, then for each free presentation $1\to R\to F\to G\to 1$ of G, $M(G, N)$ is isomorphic with the factor group $(R\cap[S,F])/[R,F]$, where S is a normal subgroup of F such that $S/R\cong N$. In particular, if $N=G$ then the Schur multiplier of (G, N) will be $M(G)=(R\cap[F,F])/[R,F]$.

We assume that the reader is familiar with the notions of the *verbal subgroup* $V(G)$, and the *marginal subgroup*

$V^*(G)$, associated with a variety of groups $\mathcal{V}$ and a group G [5] for more information on varieties of groups). Let F_∞ be the free group freely generated by the countable set $X=\{x_1,x_2,\ldots\}$ and $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups defined by the sets of laws $\mathcal{V}$ and $\mathcal{W}$, respectively. Let N be a normal subgroup of a group G, then we define $[NV^*G]$ to be the subgroup of G generated by the elements of the following set:

$$\{v(g_1,g_2,\ldots,g_in,\ldots,g_r)v(g_1,g_2,\ldots,g_r)^{-1}\mid 1\le i\le r, v\in V, g_1,\ldots,g_r\in G, n\in N\}.$$

It is easily checked that $[NV^*G]$ is the least normal subgroup T of G such that N/T is contained in $V^*(G/T)$ [6].

The first to create the generalization of the Schur multiplier to any variety of groups was Baer [7]. It is well known fact that the recent concept is useful in classifying groups into isologism classes. Leedham-Green and McKay [8] introduced the following generalized version of the Baer-invariant of a group with respect to two varieties $\mathcal{V}$ and $\mathcal{W}$.

Let G be an arbitrary group in $\mathcal{W}$ with a free presentation $1\to R\to F\to G\to 1$, in which F is a free group. Clearly, $1=W(G)=W(F)R/R$ and hence $W(F)\subseteq R$, therefore,

$$1\to R/W(F)\to F/W(F)\to G\to 1$$

is a $\mathcal{W}$-free presentation of the group G. We call

$$\mathcal{WV}M(G)=\frac{R/W(F)\cap V(F/W(F))}{[R/W(F)V^*(F/W(F))]}=\frac{W(F)(R\cap V(F))}{W(F)[RV^*F]}$$

the *generalized Baer-invariant* of the group G in $\mathcal{W}$ with respect to the variety $\mathcal{V}$. Now if N is a normal subgroup of the group G for a suitable normal subgroup S of the free group F, we have $N\cong S/R$. Then we can define the generalized Baer-invariant of the pair of groups with respect to two varieties $\mathcal{V}$ and $\mathcal{W}$ as follows:

$$\mathcal{WV}M(G,N)=\frac{R/W(F)\cap[S/W(F)V^*(F/W(F))]}{[R/W(F)V^*(F/W(F))]}=\frac{W(F)(R\cap[SV^*F])}{W(F)[RV^*F]}.$$

One may check that $\mathcal{WV}M(G, N)$ is always abelian and independent of the free presentation of G. In particular, if $\mathcal{W}$ is the variety of all groups and $N=G$ then the generalized Baer-invariant of the pair (G, N) will be

$$\mathcal{V}M(G,G)=\frac{R\cap V(F)}{[RV^*F]}=\mathcal{V}M(G),$$

which is the usual Baer-invariant of G with respect to $\mathcal{V}$ [8].

It is interesting to know the connection between the Baer-invariant of a pair of finite groups (G, N) and its factor groups with respect to the Schur-Baer variety $\mathcal{V}$. In the next section, we show that under some circumstances there are some isomorphisms among $\mathcal{V}_G$-marginal factor groups (Theorem 2.2). Also, a sufficient condition will be given such that the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of the pair of its factor groups (Theorem 2.5).

Variety $\mathcal{V}$ is called a *Schur-Baer* variety if for any group G in which

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Received November 28, 2015; **Accepted** December 21, 2015; **Published** December 23, 2015

Citation: Onsory A, Araskhan M (2015) The Generalization of the Stalling's Theorem. J Generalized Lie Theory Appl S2: 003. doi:10.4172/1736-4337.S2-003

Citation: Onsory A, Araskhan M (2015) The Generalization of the Stalling's Theorem. J Generalized Lie Theory Appl S2: 003. doi:10.4172/1736-4337.S2-003

the marginal factor group $G / V^*(G)$ is finite, then the verbal subgroup $V(G)$ is also finite. Schur [9] proved that the variety of abelian groups is a Schur-Baer variety and Baer [10] showed that a variety defined by outer commutator words carries this property. In 2002, Moghaddam et al. [11] proved that for a finite group G, $\mathcal{V}M(G)$ is finite with respect to a Schur-Baer variety $\mathcal{V}$. In the following lemma we prove similar result for the $\mathcal{WVM}(G, N)$ and $\mathcal{WVM}(G)$ with using another technique.

Lemma 1.1. *Let $\mathcal{V}$ be a Schur-Baer variety and G be a finite group in $\mathcal{W}$ with a normal subgroup N. Then there exists a group H with a normal subgroup K such that*

$$|[NV^*G]||\mathcal{WVM}(G,N)|=|[KV^*H]|<\infty.$$

In particular, $|V(G)||\mathcal{WVM}(G)|=|V(H)|<\infty.$

Proof. Let $G = F / R$ be a free presentation for the group G and S be a normal subgroup of the free group F such that $N \cong S / R$, then

$$\frac{R}{W(F)[RV^*F]} \subseteq V^*\left(\frac{F}{W(F)[RV^*F]}\right).$$

Let $H=F/W(F)[RV^*F]$ and $K=S/W(F)[RV^*F]$, then $|\frac{H}{V^*(H)}|<|G|<\infty$ and $|[KV^*H]| \le |V(H)|<\infty$. But

$$|[KV^*H]|=|\frac{W(F)[SV^*F]}{W(F)[RV^*F]}|=|\frac{W(F)[SV^*F]}{W(F)(R\cap[SV^*F])}||\frac{W(F)(R\cap[SV^*F])}{W(F)[RV^*F]}|.$$

Also, $[NV^*G]=\frac{[SV^*F]R}{R}=\frac{W(F)[SV^*F]R}{R}\cong\frac{W(F)[SV^*F]}{W(F)(R\cap[SV^*F])}$. Thus the result holds.

Stallings' Theorem

In the following lemma we present some exact sequences for the generalized Baer-invariant of a pair of groups and its factor groups.

Lemma 2.1. *Let G be a group with a free presentation $1\to R\to F\to G\to 1$ and S, T be normal subgroups of the free group F such that $T\subseteq S$, $S/R\cong N$ and $T/R\cong K$. Then the following sequences are exact:*

$$\text{(i)}\ 1\to\frac{W(F)(R\cap[TV^*F])}{W(F)[RV^*F]}\to\mathcal{WVM}(G,N)\to\mathcal{WVM}(G/K,N/K)\to\frac{K\cap[NV^*G]}{[KV^*G]}\to 1;$$

$$\text{(ii)}\ \mathcal{WVM}(G,N)\to\mathcal{WVM}(G/K,N/K)\to\frac{K}{[KV^*G]}\to\frac{N}{[NV^*G]}\to\frac{N}{[NV^*G]K}\to 1;$$

(iii) Moreover, if K is contained in $V^*(G)$, then the following sequence is exact:

$$1\to\frac{R\cap[SV^*F]}{W(F)[TV^*F]\cap[SV^*F]}\to\mathcal{WVM}(G/K,N/K)\to K\to\frac{N}{[NV^*G]}\to\frac{N}{[NV^*G]K}\to 1.$$

Proof. Considering the definition mentioned above we can conclude:

$$\mathcal{WVM}(G/K,N/K)=\frac{W(F)(T\cap[SV^*F])}{W(F)[TV^*F]} \qquad \frac{K\cap[NV^*G]}{[KV^*G]}=\frac{(T\cap[SV^*F])R}{[TV^*F]R},$$

$$\mathcal{WVM}(G,N)=\frac{W(F)(R\cap[SV^*F])}{W(F)[RV^*F]}.$$

Now one can easily check that the sequences (i) and (ii) are exact.

(iii) Using the assumption, we have $W(F)[TV^*F]\subseteq R$. Therefore, one can easily check that the following sequence is exact:

$$1\to\frac{R\cap[SV^*F]}{W(F)[TV^*F]\cap[SV^*F]}\to\frac{W(F)(T\cap[SV^*F])}{W(F)[TV^*F]}\to T/R\to\frac{S}{[SV^*F]R}\to\frac{S}{[SV^*F]T}\to 1.$$

Let N be a normal subgroup of a group G. Then we define a series of normal subgroups of N as follows:

$$N=V_0(N,G)\supseteq V_1(N,G)\supseteq V_2(N,G)\supseteq\cdots\supseteq V_n(N,G)\supseteq\cdots,$$

where $V_i(N,G)=[V_{i-1}(N,G)V^*G]$ for all $n \ge 1$. We call such a series the *lower $\mathcal{V}_G$-marginal series* of N in G. One may also define the *upper $\mathcal{V}_G$-marginal series* as in studies of Moghaddam et al. [11].

We say that the normal subgroup N of G is *$\mathcal{V}_G$-nilpotent* if it has a finite *lower $\mathcal{V}_G$-marginal series*. The shortest length of such series is called the class of *$\mathcal{V}_G$-nilpotency* of N in G. If $N = G$, then this is called *lower $\mathcal{V}$-marginal series* of G. The group G is said to be *$\mathcal{V}$-nilpotent* iff $V_n(G) = 1$, for some positive integer n [12].

Now, we want to show that under some circumstances there are some isomorphisms among $\mathcal{V}_G$-marginal factor groups. By using Lemma 2.1, we have the following Theorem, which generalizes 7.9.1 of literature of Hilton and Stammbach [13].

Theorem 2.2. *Let $f: G \to H$ be a group homomorphism and N be a normal subgroup of G and K be a normal subgroup of H such that $f(N)\subseteq K$. Suppose f induces isomorphisms $f_0: G/N\to H/K$ and $\overline{f}_1: N/[NV^*G]\to K/[KV^*H]$, and that $f_*:\mathcal{WVM}(G,N)\to\mathcal{WVM}(H,K)$ is an epimorphism. Then f induces isomorphisms $f_n: G/V_n(N,G)\xrightarrow{\cong}H/V_n(K,H)$ and $\overline{f}_n: N/V_n(N,G)\xrightarrow{\cong}K/V_n(K,H)$ for all $n \ge 0$.*

Proof. At first, we want to mention a point that for making it easier to draw the following diagrams, we would like to introduce $P_n=V_n(N,G)$ and $Q_n=V_n(K,H)$. We proceed by induction. For $n = 0$ the assertion is trivial. For $n = 1$, consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & N/[NV^*G] \arrow[r] \arrow[d, "\overline{f}_1"] & G/[NV^*G] \arrow[r] \arrow[d, "f_1"] & G/N \arrow[r] \arrow[d, "f_0"] & 1 \\
1 \arrow[r] & K/[KV^*H] \arrow[r] & H/[KV^*H] \arrow[r] & H/K \arrow[r] & 1
\end{tikzcd}$$

By the hypothesis $\overline{f}_1$ and f_0 are isomorphism, hence f_1 is an isomorphism. Assume that $n \ge 2$. By consedering Lemma 2.1(ii), we can conclude the following communicative diagram:

$$\begin{tikzcd}
WVM(G,N) \arrow[r] \arrow[d, "\alpha_1"] & WVM(G/P_{n-1},N/P_{n-1}) \arrow[r] \arrow[d, "\alpha_2"] & P_{n-1}/P_n \arrow[r] \arrow[d, "\alpha_3"] & N/[NV^*G] \arrow[r] \arrow[d, "\alpha_4"] & N/[NV^*G]P_{n-1} \arrow[r] \arrow[d, "\alpha_5"] & 1 \\
WVM(H,K) \arrow[r] & WVM(H/Q_{n-1},K/Q_{n-1}) \arrow[r] & Q_{n-1}/Q_n \arrow[r] & K/[KV^*H] \arrow[r] & K/[KV^*H]Q_{n-1} \arrow[r] & 1
\end{tikzcd}$$

Note that the naturality of the map f induces homomorphisms α_i, $i = 1,2,\ldots,5$ such that $(*)$ is commutative. By hypothesis α_1 is an epimorphism and α_4, α_5 are isomorphisms. Also, by considering the induction hypothesis and definition of the Baer-invariant of the pair of groups, α_2 is an isomorphism. Hence by five lemma of Rotman's studies [14] α_3 is an isomorphism. Now consider the following diagram and in the same way, f_n is an isomorphism.

Now we obtain the following corollary.

$$\begin{tikzcd}
1 \arrow[r] & P_{n-1}/P_n \arrow[r] \arrow[d, "\alpha_3"] & N/P_n \arrow[r] \arrow[d, "\bar{f}_n"] & N/P_{n-1} \arrow[r] \arrow[d, "\bar{f}_{n-1}"] & 1 \\
1 \arrow[r] & Q_{n-1}/Q_n \arrow[r] & K/Q_n \arrow[r] & K/Q_{n-1} \arrow[r] & 1
\end{tikzcd}$$

By the above discussion α_3 is an isomorphism and by induction of hypothesis $\bar{f}_{n-1}$ is an isomorphism, therefore, $\bar{f}_n$ is an isomorphism. Finally, by the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & N/P_n \arrow[r] \arrow[d, "\bar{f}_n"] & G/P_n \arrow[r] \arrow[d, "f_n"] & G/N \arrow[r] \arrow[d, "f_1"] & 1 \\
1 \arrow[r] & K/Q_n \arrow[r] & H/Q_n \arrow[r] & H/K \arrow[r] & 1
\end{tikzcd}$$

And the same way, f_n ia an isomorphism.

Now we obtain the following collary.

Corollary 2.3. *Let* $(f, f|): (G,N) \to (H,K)$ *are group homomorphisms satisfy the hypotheses of Theorem 2.2. Suppose further that N and K are* $\mathcal{V}_G$*-nilpotent and* $\mathcal{V}_H$*-nilpotent, respectively. Then f and f | are isomorphisms.*

Proof. The assertion follows from Theorem 2.2 and the remark that there exists $n \geq 0$ such that $V_n(N,G) = \{1\}$ and $V_n(K,H) = \{1\}$.

Now, we have the following theorem, which is a generalization of Stalling's theorem [15].

Theorem 2.4. *Let* $\mathcal{V}$ *be a variety of groups and* $f : G \to H$ *be an epimorphism. Let N be a* $\mathcal{V}_G$*-nilpotent normal subgroup of G and K be a normal subgroup of H such that f (N) = K. If* $\ker f \subseteq [NV^*G]$ *and* $\mathcal{WVM}(H, K)$ *is trivial, then f and f | are isomorphisms.*

Proof. Put $M = \ker f$, then $\frac{N}{[NV^*G]} \cong \frac{K}{[KV^*H]}$, $\frac{G}{N} \cong \frac{H}{K}$ and $\frac{V_n(N,G)M}{M} = V_n(K,H)$ for all $n \geq 0$. Now the result follows from Corollary 2.3.

Finally, a sufficient condition will be given such that the order of the generalized Baer-invariant of a pair of finite groups divides the order of the generalized Baer-invariant of the pair of its factor groups with respect to two varieties of groups. Let $\psi : E \to G$ be an epimorphism such that $\ker \psi \subseteq V^*(E)$. We denote by $(WV^*)^*(G)$ the intersection of all subgroups of the form $\psi(V^*(E))$. Clearly, $(WV^*)^*(G)$ is a characteristic subgroup of G which is contained in $V^*(G)$. In particular, if $\mathcal{W}$ is the variety of all groups and $\mathcal{V}$ is a variety of abelian groups then this subgroup is denoted by $Z^*(G)$ as in literature of Karpilovsky [2].

Now using the above concept we have the following Theorem.

Theorem 2.5. *Let K be a normal subgroup of G contained in* $N \cap (WV^*)^*(G)$. *Then*

$|\mathcal{WVM}(G,N)|$ *divides* $|\mathcal{WVM}(G/K, N/K)|$.

Proof. By theorem 3.2 of Neumann [5], natural homomorphism $\mathcal{WVM}(G) \to \mathcal{WVM}(G/K)$ will be a monomorphism. Now the following commutative diagram

$$\begin{tikzcd}
WVM(G,N) \arrow[r, "\subseteq"] \arrow[d] & WVM(G) \arrow[d] \\
WVM(G/K, N/K) \arrow[r, "\subseteq"] & WVM(G/K)
\end{tikzcd}$$

implies that the natural homomorphism $\mathcal{WVM}(G,N) \to \mathcal{WVM}(G/K, N/K)$ is also a monomorphism. Thus Lemma 1.2 (i) implies that $\mathcal{WVM}(G, K)$ is trivial. Now we have $|\mathcal{WVM}(G/K, N/K))| = |K \cap [NV^*G]||\mathcal{WVM}(G,N)|$, which completes the result.

Acknowledgment

The authors wish to thank Yazd Branch, Islamic Azad University for its support of research project under the title the Generalization of the Stalling's theorem.

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Citation: Onsory A, Araskhan M (2015) The Generalization of the Stalling's Theorem. J Generalized Lie Theory Appl S2: 003. doi:10.4172/1736-4337.S2-003

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, University of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications

Nadjafikhah and Pourrostami, J Generalized Lie Theory Appl 2015, S2
http://dx.doi.org/10.4172/1736-4337.S2-004

Research Article Open Access

Self-adjointness, Group Classification and Conservation Laws of an Extended Camassa-Holm Equation

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Abstract

In this paper, we prove that equation $E \equiv u_t - u_{x^2 t} + u_x f(u) - a u_x u_{x^2} - b u u_{x^3} = 0$ is self-adjoint and quasi self-adjoint, then we construct conservation laws for this equation using its symmetries. We investigate a symmetry classification of this nonlinear third order partial differential equation, where *f* is smooth function on *u* and *a*, *b* are arbitrary constans. We find Three special cases of this equation, using the Lie group method.

Keywords: Lie symmetry analysis; Self-adjoint; Quasi self-adjoint; Conservation laws; Camassa-Holm equation; Degas peris-Procesi equation; Fornberg whitham equation; BBM equation

Introduction

A new procedure for constructing conservation laws was developed by Ibragimov [1]. For Camassa-Holm equation are calculated in studies of Ibragimov, Khamitova and Valenti [2]. In this paper, we study the following third-order nonlinear equation

$$E \equiv u_t - u_{x^2 t} + u_x f - a u_x u_{x^2} - b u u_{x^3} = 0, \tag{1}$$

and we show that this equation is self-adjoint and quasi self-adjont. Therefore we find Lie symmetries and conservation laws. There are three cases to consider: 1) $b \neq 0$, a = arbitrary constant, 2) $b = 0$, $a \neq 0$, and 3) $b = 0$, $a = 0$. Clarkson, Mansfield and Priestly [3] are concerned with symmetry reductions of the non-linear third order partial differential equation given by $u_t - \in u_{x^2 t} + (k-u)u_x - u u_{x^3} - \beta u_x u_{x^2} = 0$, where $\in$, k, and β are arbitrary constants. Symmetry classification and conservation laws for higher order Camassa-Holm equation are calculated in framework of Nadjafikhah and Shirvani-Sh [4].

The special cases of (1) are:

Camassa-Holm (CH) equation $u_t - u_{x^2 t} + (k+3u)u_x = u u_{x^3} + 2u_x u_{x^2}$, k-arbitrary (real), describing the unidirectional propagation of shallow water waves over a flat bottom (let $f = k + 3u$, $a = 2$, $b = 1$ in (1).

Degas peris-Procesi (DP) equation $u_t - u_{x^2 t} + (k+4u)u_x = u u_{x^3} + 3u_x u_{x^2}$, k-arbitrary (real), is another equation of this class (let $f = k + 4u$, $a = 3$, $b = 1$ in (1).

Fornberg Whitham (FW) equation $u_t - u_{x^2 t} + (1+u)u_x = u u_{x^3} + 3u_x u_{x^2}$, is another equation of this class (let $f = 1 + u$, $a = 3$, $b = 1$ in (1)).

BBM equation $u_t - u_{x^2 t} + u_x + (u u_x) = 0$, is another equation of this class (let $f = 1 + u$, $a = 0$, $b = 0$ in (1)).

Preliminaries

In this section, we recall the procedure in literature of Ibragimov [1]. Let us introduce the formal Lagrangian

$$L \equiv vE, \tag{2}$$

where $v = v(t, x)$ is a new dependent variable.

We define the adjoint equation by $E^* \equiv \frac{\delta L}{\delta u} = 0$. Here

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}} - D_i D_j D_k \frac{\partial}{\partial u_{ijk}} + \cdots \qquad i,j,k = 1,2,$$

is the variational derivative and D_i is the operator of total diferentiation.

An equation $E = 0$ is said to be self-adjoint [5] if the equation obtained from the adjoint equation by substitution $v = u$ is identical with the original equation.

An equation $E = 0$ is said to be quasi- self-adjoint [5] if there exists a function $v = \varphi(u)$, $\varphi'(u) \neq 0$ such that $E^*|_{v=\varphi(u)} = \lambda E$ with an undetermined coefficient λ. Eq.(1) is said to have a nonlocal conservation law if there exits a vector $C = (C^1, C^2)$ satisfying the equation

$$D_t(C^1) + D_x(C^2) = 0, \tag{3}$$

on any solution of the system of differential equations comprising (E) and the adjoint equation (E^*). We say that orginal equation has a local conservation law if (3) is satisfied on any solution of Eq.(1). In studies of Ibragimov [1], the conserved vector associated with the Lie point symmetry $v = \xi^1(x,t,u)\partial_x + \xi^2(x,t,u)\partial_t + \phi(x,t,u)\partial_u$ is obtained by the following formula :

$$C^i = \xi^i L + W\left[\frac{\partial L}{\partial u_i} - D_j\left(\frac{\partial L}{\partial u_{ij}}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}}\right)\right]$$

$$+ D_j(W)\left[\frac{\partial L}{\partial u_{ij}} - D_k\left(\frac{\partial L}{\partial u_{ijk}}\right)\right] + D_j D_k(W)\frac{\partial L}{\partial u_{ijk}}, \tag{4}$$

where $i, j, k = 1,2$ and $W = \phi - \xi^i u_i$. (Here ∂_x means $\frac{\partial}{\partial x}$).

We recall the general procedure for determining symmetries for an arbitrary system of partial differential equations [6]. Let us consider the general system of a nonlinear system of partial differential equations of order n, containing p independent and q dependent variables is given as follows

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l, \tag{5}$$

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Received April 21, 2015; **Accepted** December 22, 2015; **Published** December 24, 2015

Citation: Nadjafikhah M, Pourrostami N (2015) Self-adjointness, Group Classification and Conservation Laws of an Extended Camassa-Holm Equation. J Generalized Lie Theory Appl S2: 004. doi:10.4172/1736-4337.S2-004

involving $x=(x^1,\cdots,x^p)$, $u=(u^1,\cdots,u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We consider a one-parameter Lie group of transformations acting on the variables of system (5): $\overline{x}_i = x^i + \in \xi^i(x,u) + O(\in^2)$, $\overline{u}_j = u^j + \in \phi^j(x,u) + O(\in^2)$, where $i=1,\cdots,p$, $j=1,\cdots,q$. ξ^i, φ^j are the infinitesimal of the transformations for the independent and dependent variables, respectively, and $\in$ is the transformation parameter. We consider the general vector field v as the infinitesimal generator associated with the above group $v=\sum_{i=1}^p \xi^i(x,u)\partial_{x^i} + \sum_{j=1}^q \phi^j(x,u)\partial_{u^j}$. A symmetry of a differential equation is a transformation, which maps solutions of the equation to other solutions. The invariance of the system (5) under the infinitesimal transformation leads to the invariance conditions. (Theorem 2.36 of studies of Olver [6], Theorem 6.5 of literature of Olver [7]).

$$v^n[\Delta_\nu(x,u^n)]=0,\quad \Delta_\nu(x,u^n)=0,\quad \nu=1,\cdots,r, \tag{6}$$

where v^n is called the n^{th} order prolongation of the infinitesimal generator given by $v^n = v + \sum_{j=1}^q \sum_k \phi_k^j(x,u^{(n)})\partial_{u_k^j}$, where $k=(i_1,\cdots i_\alpha)$, $1 \le i_\alpha \le p$, $1 \le \alpha \le n$, and the sum is over all k's of order $0 < \#k \le n$. If $\#k = \alpha$, the coefficent ϕ_k^j of $\partial_{u_k^j}$, will depend only on α'th and lower order derivatives of u and $\phi_j^k(x,u^n) = D_k(\phi_j - \sum_{i=1}^p (\xi^i u_i^j)) + \sum_{i=1}^p \xi^i u_{k,i}^j$, where $u_i^j := \partial u^j / \partial x^i$ and $u_{k,i}^j := \partial u_k^j / \partial x^i$.

Adjoint Equation and Classical Symmetry Method

Formal Lagrangian for Eq. (1) is

$$L = vE = v[u_t - u_{x^2t} + u_x f - a u_x u_{x^2} - b u u_{x^3}]. \tag{7}$$

Therefore, the adjoint equation E^* to Eq. (1) is

$$f v_x + v_t + a u_{x^2} v_x + a u_x v_{xx} = 3b u_{x^2} v_x + 3b u_x v_{xx} + b u v_{xxx} + v_{xxt}. \tag{8}$$

Upon setting $v = u$ it becomes

$$u_t = u_{x^2t} - u_x f - 2a u_x u_{x^2} + 6b u_x u_{x^2} + b u u_{x^3}.$$

Hence, Eq. (1) is self-adjoint if and only if it has the form

$$a = 2b. \tag{9}$$

Consider again Eq. (1), and substitute

$$v=\varphi(u),\qquad v_t=\varphi' u_t,$$

$$v_x=\varphi' u_x,\qquad v_{xx}=\varphi' u_{x^2}+\varphi'' u_x^2,$$

$$v_{xxx}=\varphi' u_{x^3}+3\varphi'' u_x u_{x^2}+\varphi''' u_x^3,$$

$$v_{xxt}=\varphi' u_{x^2t}+\varphi'' u_t u_{x^2}+\varphi''' u_x^2 u_t+2\varphi'' u_x u_{xt},$$

in the adjoint equation (8), then

$$\begin{aligned}&-f\varphi' u_x - \varphi' u_t - 2a\varphi' u_x u_{x^2} - a\varphi'' u_x^3 + 6b\varphi' u_x u_{x^2}\\ &+3b\varphi'' u_x^3 + b\varphi' u u_{x^3} + 3b\varphi'' u u_x u_{x^2} + b\varphi''' u u_x^3\\ &+\varphi' u_{x^2t} + \varphi'' u_{x^2} u_t + \varphi''' u_x^2 u_t + 2\varphi'' u_x u_{xt}\\ &=\lambda(u_t - u_{x^2t} + u_x f - a u_x u_{x^2} - b u u_{x^3}).\end{aligned}$$

Hence, Eq. (1) is quasi self-adjoint if and only if it has the form

$$a = 2b,\ v = -\lambda u + \varepsilon \tag{10}$$

In this section, we will perfom Lie group method for Eq. (1) on $(x^1 = x,\ x^2 = t,\ u^1 = u)$, $(\tilde{x},\tilde{t},\tilde{u}) = (x,t,u) + \in(\xi(x,t,u),\tau(x,t,u),\phi(x,t,u)) + O(\in^2)$, where $\varepsilon \le 1$ the group parameter and $\xi^1=\xi$, $\xi^2=\tau$ and $\phi^1=\phi$ are the infinitesimals of the transformations for the independent and dependent variables respectively. The associated vector fields is of the form $v=\xi(x,t,u)\partial_x+\tau(x,t,u)\partial_t+\phi(x,t,u)\partial_u$ and the third porolongation of v is the vector field

$$v^{(3)} = v + \phi^x\partial_{u_x} + \phi^t\partial_{u_t} + \phi^{x^2}\partial_{u_{x^2}} + \phi^{xt}\partial u_{xt} + \cdots + \phi^{ttt}\partial u_{ttt},$$

with coefficent

$$\phi^k = D_k(\phi - \sum_{i=1}^2 \xi^i u_i^j) + \sum_{i=1}^2 \xi^i u_{k,i}, \tag{11}$$

where D_k is the total derivative with respect to independent variables. The invariance condition (6) for Eq. (1) is given by,

$$v^{(3)}[u_t - u_{x^2t} + u_x f - a u_x u_{x^2} - a u u_{x^3}] = 0, \tag{12}$$

whenever $E = 0$. The condition (12) is equvalent to

$$\phi^t - b u_{x^3}\phi + (f - a u_{x^2})\phi^x - a u_x \phi^{x^2} - b u \phi^{x^3} - \phi^{x^2t} = 0, \tag{13}$$

whenever $E = 0$. Substituting (11) into (13), yields the determining equations. There are three cases to consider:

a and $b \neq 0$ are arbitrary constants

In this case, complete set of determining equation is:

$$\xi_u = 0, \tag{14}$$

$$\tau_u = 0, \tag{15}$$

$$\tau_x = 0, \tag{16}$$

$$\phi_{u^2} = 0, \tag{17}$$

$$a\phi_{ux^2} + a\phi_u + a\tau_t - 3a\xi_x = 0, \tag{18}$$

$$\xi_{x^2} - 2\phi_{ux} = 0, \tag{19}$$

$$3bu\phi_{ux} + a\phi_x - 3b\xi_{x^2}u + 2\xi_{xt} + \phi_{ut} = 0, \tag{20}$$

$$2\xi_x - \phi_{ux^2} = 0, \tag{21}$$

$$\xi_t - b\phi_x - \phi_{ux^2} - bu\tau_t + 3bu\xi_x = 0, \tag{22}$$

$$a\xi_{x^2}u - 2a\phi_{ux} = 0, \tag{23}$$

$$\xi_{x^2t} + bu\xi_{x^3} + \phi f_u + \phi_{ux^2} f + f\xi_x + \tau_t f = a\phi_{x^2} + \xi_t + 2\phi_{xtu} + 3bu\phi_{ux^2}, \tag{24}$$

$$-bu\phi_{x^3} + \phi_t + \phi_x f - \phi_{x^2t} = 0. \tag{25}$$

With substituting (14) – (17) into (18) – (23) we have

$$\phi = c_1 + \frac{1}{b}\alpha'(t),\quad \tau = -c_1 t + c_2,\quad \xi = \alpha(t). \tag{26}$$

With substituting (26) into (24) – (25) we have

$$f = -1 + K(bu+1), \tag{27}$$

where c_1, c_2 and K are arbitrary constants. With substituting (27) into determining system, we have

$$\phi = \frac{-c_1(bu+1)}{b},\quad \tau = c_1 t + c_2,\quad \xi = -c_1 t + c_3,$$

where c_i, $i = 1,2,3$ are arbitrary constants.

Theorem 3.1.1. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1 = -t\partial_x + t\partial_t - \frac{(bu+1)}{b}\partial_u,\quad v_2 = \partial_t,\quad v_3 = \partial_x.$$

We want to construct the conservation law associated with the symmetry

$$v_1 = -t\partial_x + t\partial_t - \frac{(bu+1)}{b}\partial_u.$$

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We have

$$W=-u-\frac{1}{b}-tu_t+tu_x. \tag{28}$$

The right-hand side of (4) is written

$$\begin{aligned} C^1&=W(v-D_x^2(v))+(D_xW)(D_xv)-D_x^2(W)v,\\ C^2&=W[vf-avu_{x^2}+D_x(avu_x)-D_x^2(buv)-2D_tD_x(v)]\\ &+D_x(W)[-avu_x+D_x(buv)+D_t(v)]+D_t(W)[D_x(v)]\\ &-2D_xD_t(W)[v]-D_x^2(W)[buv]. \end{aligned} \tag{29}$$

We eliminate the term ξ^iL since the Lagrangian L is equal to zero on solution of Eq.(1). Substituting in (29), the expression (7) for L and (28) for W, we obtain

$$\begin{aligned} C^1&=-uv-\frac{1}{b}v-tu_tv+tu_xv+uv_{xx}+\frac{1}{b}v_{xx}+tu_tv_{xx}-tu_xv_{xx}\\ &-u_xv_x-tu_{xt}v_x+tv_xu_{x^2}+u_{x^2}v+tu_{x^2t}v-tu_{x^3}v, \end{aligned} \tag{30}$$

and

$$\begin{aligned} C^2&=-u(vf-bvu_{x^2}-buv_{xx}-2v_{xt})\\ &-(b^{-1}(vf-bvu_{x^2}-buv_{xx}-2v_{xt}))\\ &-tu_t(vf-bvu_{x^2}-buv_{xx}-2v_{xt})\\ &+tu_x(vf-bvu_{x^2}-buv_{xx}-2v_{xt})-u_x(buv_x-bvu_x+v_t)\\ &-(tu_{xt})(buv_x-bvu_x+v_t)+(tu_{x^2})(buv_x-bvu_x+v_t)-2u_tv_x\\ &-tu_{t^2}v_x+u_xv_x+tu_{xt}v_x+buvu_{x^2}+tbuvu_{x^2t}-tbuvu_{x^3}\\ &+4u_{xt}v+2tu_{xt^2}v-2u_{x^2}v-2u_{x^2t}v. \end{aligned} \tag{31}$$

We can eliminate u_t by using Eq.(1) and then substitute in (30) and (31) the expression $v = u$, therefore arrive at the conserved vector with the following components:

$$\begin{aligned} C^1&=\frac{-1}{b}(t(2bu_xu_{x^2}+buu_{x^3}-u_xf+u_{x^2t})ub\\ &-t(2bu_xu_{x^2}+buu_{x^3}-u_xf+u_{x^2t})u_{x^2}b-tu_xub\\ &+tu_{xt}u_xb+tu_{x^3}ub-tu_{x^2t}ub+u^2b-2uu_{x^2}b+u_x{}^2b+u-u_{x^2}), \end{aligned} \tag{32}$$

$$\begin{aligned} C^2&=-u(uf-2buu_{xx}-2u_{xt})-(b^{-1}(uf-2buu_{xx}-2u_{xt}))\\ &-t(2bu_xu_{x^2}+buu_{x^3}-u_xf+u_{x^2t})(uf-2buu_{xx}-2u_{xt})\\ &+tu_x(uf-2buu_{xx}-2u_{xt})-u_x(u_t)-tu_{xt}(u_t)\\ &+(tu_{x^2})(u_t)-2(2bu_xu_{x^2}+buu_{x^3}-u_xf+u_{x^2t})u_x\\ &-tu_{t^2}u_x+u_{xt}+tu_{xt}u_x+bu^2u_{x^2}+tbu^2u_{x^2t}-tbu^2u_{x^3}+4u_{xt}u\\ &+2tu_{xt^2}u-2uu_{x^2}-2u_{x^2t}u. \end{aligned}$$

Where $f=-1+K(bu+1)$.

a is an arbitrary nonzero constant and $b = 0$.

In this case Eq.(1) is not self adjoint because $a \neq 2b$. Complete set of determining equation is:

$$\phi_{uu}=0, \tag{33}$$

$$\xi_u=0, \tag{34}$$

$$\xi_t=0, \tag{35}$$

$$\tau_u=0, \tag{36}$$

$$\tau_x=0, \tag{37}$$

$$3a\xi_x=a\phi_{ux^2}+a\tau_t-a\phi_u, \tag{38}$$

$$\phi_{ut}+a\phi_x=0, \tag{39}$$

$$-2\phi_{ux}+\xi_{x^2}=0, \tag{40}$$

$$2\xi_x-\phi_{ux^2}=0, \tag{41}$$

$$a\xi_{x^2}-2a\phi_{ux}=0, \tag{42}$$

$$\tau_tf+\phi f_u+\phi_{ux^2}f=2\phi_{uxt}+f\xi_x+a\phi_{x^2}, \tag{43}$$

$$\phi_t+\phi_xf=\phi_{x^2t}. \tag{44}$$

Now, by considering Eq. (33) – (42) it is not to hard to find that the components ξ, τ and φ of infinitesimal generators become

$$\phi=u\frac{dF_1(t)}{dt}-\frac{x}{a}\frac{d^2F_1(t)}{dt^2}+F_2(t),\quad \tau=-F_1(t)+c_2,\quad \xi=c_1. \tag{45}$$

To find complete solution of the above system, we consider Eq. (43) and $l = \dim \mathrm{Spam}_{\mathbb{R}}\{f_u, f, 1\}$. Three general cases are possible:

3.2.i) $l = 1$, then f = constant;

3.2.ii) $l = 2$, then $f_u = \alpha f + \beta$;

3.3.iii) $l = 3$, then $\alpha f_u + \beta f + \gamma \neq 0$, $\alpha \neq 0$.

Case 3.2.i). With substituting f = constant in determining system (33)-(44), we have $\varphi = c_1$, $\tau = c_2$, $\xi = c_3$, where c_i, $i = 1,2,3$ are arbitrary constants.

Theorem 3.2.1. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1=\partial_x,\quad v_2=\partial_t,\quad v_3=\partial_u.$$

Case 3.2.ii). With integrating from $f_u = \alpha f + \beta$ with respect to u, we obtain

$$f=\frac{-\beta}{\alpha}+Ce^{\alpha u}, \tag{46}$$

where C is an integrating constant. With substituting (46) into Eq. (43)-(44) and Eq. (45), we have

$$\xi=c_1,\qquad \tau=-c_1t,\qquad \phi=\frac{c_1(C\alpha-e^{-\alpha u}\beta)}{C\alpha^2}. \tag{47}$$

Theorem 3.2.2. *Infinitesimal generator of every one parameter Lie group of point symmetries in this case is:*

$$v=\partial_x-t\partial_t+\frac{C\alpha-e^{-\alpha u}\beta}{C\alpha^2}\partial_u. \tag{48}$$

Case 3.2.iii). The Eq. (43) leads to $\varphi = 0$, $\tau = c_1$, $\xi = c_2$.

Theorem 3.2.3. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1=\partial_t,\quad v_2=\partial_x.$$

$b = 0$, $a = 0$.

Complete set of determining equation is

$$\xi_u=0, \tag{49}$$

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$$\xi_t = 0, \tag{50}$$

$$\tau_u = 0, \tag{51}$$

$$\tau_x = 0, \tag{52}$$

$$\phi_{ut} = 0, \tag{53}$$

$$\phi_{uu} = 0, \tag{54}$$

$$\phi_{ux^2} = 2\xi_x, \tag{55}$$

$$2\phi_{ux} = \xi_{x^2}, \tag{56}$$

$$\phi_t + f\phi_x = \phi_{x^2 t}, \tag{57}$$

$$f\tau_t + f\xi_x + \phi f_u = 0. \tag{58}$$

To find a complete solution of the above system we consider Eq. (58) and with assumption $f/f_u \neq 0$ we rewrite:

$$\phi = \frac{-f}{f_u}(\tau_t + \xi_x). \tag{59}$$

Two general cases are possible:

3.3.*i*) $\frac{f}{f_u} = c$, 3.3.*ii*) $\frac{f}{f_u} = h(u)$,

where c is constant.

Case 3.3.i).

With integrating from $f/f_u \neq c$ with respect to u, we have

$$f = Le^{u/c}, \tag{60}$$

where L is an integrating constant. Then the Eq. (58) reduce to

$$\phi = -c(\tau_t + \xi_x). \tag{61}$$

With substituting Eq. (61) into determining equation, we have

$$\xi = c_1, \quad \tau = c_2 t + c_3, \quad \phi = -cc_2, \tag{62}$$

where c_i, $i = 1,2,3$ are arbitrary constants.

Theorem 3.3.1. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1 = t\partial_t - c\partial_u, \quad v_2 = \partial_t, \quad v_3 = \partial_x.$$

We want to construct the conservation law associated with the symmetry

$$v_1 = t\partial_t - c\partial_u.$$

We have

$$W = -c - tu_t. \tag{63}$$

The right-hand side of (4) is written

$$C^1 = W(v - v_{xx}) + (D_x(W))[v_x] - D_x^2(W)[v], \tag{64}$$

$$C^2 = W[vf - 2v_{xt}] + D_x(W)[v_t] + D_t(W)[v_x] - 2D_xD_t(W)[v]. \tag{65}$$

Substituting in (64) and (65), the expression (7) for L and (63) for W, we obtain

$$C^1 = -cv + cv_{xx} - tvu_t + tv_{xx}u_t - tv_x u_{tx} + tvu_{txx}, \tag{66}$$

$$C^2 = -cvf - u_t v_x + 2cv_{xt} - tu_{xt}v_t$$
$$-tv_x u_{t^2} - tvfu_t + 2vu_{tx} + 2tu_{xt^2}v + 2tu_t v_{xt}. \tag{67}$$

We can eliminate u_t by using Eq. (1) and obtain

$$C^1 = -cv + cv_{xx} + tvfu_x + tv_{xx}u_{x^2 t} - tfv_{xx}u_x - tv_x u_{tx}, \tag{68}$$

$$C^2 = -u_{x^2 t}v_x + fu_x v_x + 2cv_{xt} - tu_{xt}v_t - tv_x u_{t^2} - tvfu_{x^2 t} + tvf^2 u_x$$
$$+2vu_{tx} + 2tu_{xt^2}v - cvf + 2tv_{xt}u_{xxt} - 2tfu_x v_{xt}. \tag{69}$$

Now, we substitute in (68) and (??) the expression $v = u$, therefore arrive at the conserved vector with the following components:

$$C^1 = -cu + cu_{x^2} + tufu_x + tu_{x^2}u_{x^2 t} - tfu_{x^2}u_x - tu_x u_{tx}, \tag{70}$$

$$C^2 = -cuf - u_{x^2 t}u_x + fu_x^2 + 2cu_{xt} - tu_{xt}u_t - tu_x u_{t^2} + 2tu_{xt}u_{xxt}$$
$$-2ftu_x u_{xt} - tufu_{x^2 t} + tuf^2 u_x + 2uu_{tx} + 2tu_{xt^2}u, \tag{71}$$

where $f = Le^{u/c}$.

Case 3.3.ii). By considering Eq. (49) – (54), we find that the components ξ, τ and φ are $\xi = \xi(x)$, $\tau = \tau(t)$ and $\phi = A(x)u + B(x,t)$. By considering Eq. (55) and (56) we have

$$\xi = c_1 \exp 2x + c_2 \exp -2x + c_3,$$

$$A(x) = c_1 \exp 2x - c_2 \exp -2x + c_4.$$

By considering Eq. (57) we have

$$\tau = ft^2(2c_1 \exp 2x + 2c_2 \exp -2x) + c_5 t + c_6,$$

where c_i, $i = 1..6$ are arbitrary constants.

From the following identity:

$$A(x)u + B(x,t) = \frac{-f}{f_u}(\tau_t + \xi_x),$$

we find that $c_1 = c_2 = 0$ and $\phi = -(f/f_u)c_5$. Hence we have two particular cases:

$$\frac{f}{f_u} = Ku, \qquad \frac{f}{f_u} \neq Ku = g(u),$$

where K is an arbitrary nonzero constant. For the first case, we have

$$\xi = c_3, \quad \tau = c_5 t + c_6, \quad \phi = -Kuc_5,$$

and for the second case, we have

$$\xi = c_3, \quad \tau = c_6, \quad \phi = 0.$$

Theorem 3.2. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case, when* $f/f_u = Ku$ *are*

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = t\partial_t - u\partial_u,$$

and when $f/f_u \neq Ku = g(u)$ are

$$v_1 = \partial_x, \quad v_2 = \partial_t,$$

where K is an arbitrary nonzero constant.

To construct the conservation law associated with the symmetry $v = t\partial_t - u\partial_u$, we find that $W = -u - tu_t$. Therefore, we have the conserved vector with the following components:

$$C^1 = -u^2 + uu_{xx} - tuu_{xxt} + tfuu_x + tu_{xx}u_{xxt}$$
$$-tfu_x u_{xx} - u_{x^2} - tu_x u_{xt} + uu_{xx} + tuu_{xxt},$$

$$C^2 = -u^2 f - tufu_{xxt} + tuf^2 u_x + 2uu_{xt} + 2tu_{xxt}u_{xt} - 2ftu_{xt}u_x$$
$$-u_x u_t - tu_{xt}u_t + 4u_{xt}u + 2tu_{ttx}u - 2u_{xxt}u_x + 2fu_x^2 - 2u_{tt}u_x,$$

where $f/f_u \neq Ku = g(u)$.

Acknowledgements

The authors wish to express their sincere gratitude to Prof. N.H. Ibragimov for his useful advise and suggestions and helpful comments.

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Citation: Nadjafikhah M, Pourrostami N (2015) Self-adjointness, Group Classification and Conservation Laws of an Extended Camassa-Holm Equation. J Generalized Lie Theory Appl S2: 004. doi:10.4172/1736-4337.S2-004

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, Unversity of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications
ISSN: 1736-4337

Ndoune and Bouetou, J Generalized Lie Theory Appl 2015, S2
http://dx.doi.org/10.4172/1736-4337.S2-005

Research Article Open Access

On Representations of Bol Algebras

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Abstract

In this paper, we introduce the notion of representation of Bol algebra. We prove an analogue of the classical Engel's theorem and the extension of Ado-Iwasawa theorem for Bol Algebras. We study the category of representations of Bol algebras and show that it is a tensor category. In the case of regular representations of Bol algebras, we give a complete classification of them for all two-dimensional Bol algebras.

Keywords: Bol algebra; Lie triple System; Non-associative algebras; Jordan superalgebras; Nilpotent representation

Introduction

It is well known that the algebraic systems which characterize locally a totally geodesic subspace is a Lie triple system [1-3]. A Bol algebra is realized by equipping Lie triple System with an additional binary skew operation which satisfies a pseudo-differentiation property [4,5]. A morphism of Bol algebras is a linear map which preserves the ternary and the binary operations. More generally, the algebraic structures which characterize locally Bol loops are Bol algebras [6]. Until now, the representations of these algebras have not been studied. Since the representations of Lie algebras and Lie groups have natural connection with particulars physics, we claim that the representations of Bol algebras should lead with the physical applications. More precisely, in physics the representations of Bol algebras will be useful for the description of invariant properties of physical systems. and the concomitant conservation laws as a result. In literature of Mostovoy and Pérez-Izquierdo [7], it is shown that, Malcev algebras and Lie triple systems are particular subclasses of Bol algebras. The representations of Malcev algebras can be found studies of Kuz'min [8], and those of Lie triple systems were constructed by Hodge and Parshall [9], Bertrand, et al. [10]. Now, there already exists some representations of other classes of non-associative algebras; the case of alternative algebras was constructed by Schafer [11], the one of Leibniz algebras by Kolesnikov [12] and for Jordan superalgebras, the representations was given by Consuelo and Zelmanov [13].

Let $\mathfrak{B}$ be a Bol algebra over a field K of characteristic zero, a representation of Bol algebra $\mathfrak{B}$ on a K-vector space V is a triplet of maps (ρ,δ,Δ) which respect some conditions which will be given later in the paper.

Our first main result is the following.

Theorem 1.1. *Let $\mathfrak{B}$ be a finite dimensional Bol algebra over a field K and $\mathcal{R}$ consist of nilpotent representations of Bol algebra $\mathfrak{B}$ in a finite dimensional space V. Then there exists a vector $v \in V$, $v \neq 0$ such that $(\rho,\delta,\Delta)(v)=0$ for all $(\rho,\delta,\Delta)\in\mathcal{R}$.*

We agree that the image of any vector v of V by the operator (ρ,δ,Δ) is given by $(\rho,\delta,\Delta)(v)=(\rho(v_1),\delta(v_2),\Delta(v_3))$, where $v=(v_1,v_2,v_3)\in\mathfrak{B}^3$.

We define also the regular representations and the adjoint representations of Bol algebras. As an easy consequence, we show that if any representation of Bol algebra is nilpotent, then its adjoint representation is also nilpotent.

We are also interested by the question of the extension theorem of Ado-Iwasawa for Bol algebras. Pérez-Izquierdo established the existence of a Poincaré-Birkhoff-Witt type basis for a universal envelope of Bol algebra [5]. The above result allows us to interest ourselves to an extension of Ado-Iwasawa theorem for Bol algebra. let A be an alternative algebra, the the generalized right alternative nucleus is the algebra $RN_{alt}(A)$ defined by $RN_{alt}(A)=\{a\in A/(x,a,y)=-(x,y,a)\}$. We then give our second theorem.

Theorem 1.2. *Let $\mathfrak{B}$ be a finite-dimensional right Bol algebra over a field of characteristic different to 2 and 3. Then there exists a unital finite-dimensional algebra A and a monomorphism of Bol algebras $\mathfrak{B}\to RN_{alt}(A)$.*

The analogue of our second result above was established for Malcev algebras framed by Pérez-Izqquierdo and Shestakov [14]. The collection of all representations of Bol algebra and the morphisms between them form a category, named the category of representations of Bol algebras $Rep(\mathfrak{B})$. One can view a representation of Bol algebra as a B-module analogously as in literature of Consuelo and Zelmanov [13] in the case of Jordan superalgebras. One can understand also the representations of Bol algebras in term of matrices with sweet properties. The investigation between the category $Rep(\mathfrak{B})$ and the category of left $U(\mathfrak{B})$ -modules, where $U(\mathfrak{B})$ is the universal enveloping algebra of $\mathfrak{B}$, endowed with its bialgebra structure, leads us to our third main theorem.

Theorem 1.3. *The category of representations of Bol algebra $Rep(\mathfrak{B})$ is equivalent to the category of representations of its universal enveloping algebra $Rep(U(\mathfrak{B}))$.*

The paper is organized as follows: We introduce in section 2 the notion of representations of Bol algebra. In section 3 we establish the Engel's theorem for Bol algebras. In section 4 an extension of Ado-Iwasawa theorem to Bol algebras is proved. Finally in section 5, we present the category of representations of Bol algebras and show that

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Received December 02, 2015; **Accepted** December 13, 2015; **Published** December 15, 2015

Citation: Ndoune N, Bouetou TB (2015) On Representations of Bol Algebras. J Generalized Lie Theory Appl S2: 005. doi:10.4172/1736-4337.S2-005

Citation: Ndoune N, Bouetou TB (2015) On Representations of Bol Algebras. J Generalized Lie Theory Appl S2: 005. doi:10.4172/1736-4337.S2-005

it is equivalent to the category of left modules under its universal enveloping algebra. As immediate consequence, we show the category $Rep(\mathfrak{B})$ is a tensor category. We end the section by given a complete classification of regular representations of two-dimensional Bol algebras.

Bol Algebras and their Representations

Bol algebras were introduced in differential geometry to study smooth Bol loops [6,15,16]. A *right loop* is a set $\mathcal{Q}$, together with a binary operation $(a,b)\mapsto a\cdot b$, such that for any b in $\mathcal{Q}$, the right multiplication operator $R_b : x\mapsto x\cdot b$ is bijective, and there exists an element $\varepsilon\in\mathcal{Q}$, such that $\varepsilon\cdot b=b$ for any b in $\mathcal{Q}$. The dual definition gives rise to a left Bol loop. In case that $\langle\mathcal{Q},\cdot,\varepsilon\rangle$ is both left and right loop then it is called a loop with identity element ε.

A *right smooth loop* $\mathcal{M}$ is a right loop equipped with a structure of smooth manifold, that is the map $(a,b)\mapsto a\cdot b$ and R_b^{-1} are smooth, [15,16]. Since groups are particular loops, so the Lie groups are particular cases of smooth loops. In scientific literature, many classes of loops are known: homogeneous loops, Moufang loops, Bol loops, Kikkawa loops among others.

A *right Bol loop* $\langle\mathcal{Q},\cdot,\varepsilon\rangle$ is a right loop that satisfies the right Bol identity

$$x\cdot((a\cdot y)\cdot a)=((x\cdot a)\cdot y)\cdot a$$

for all a, x, y in $\mathcal{Q}$. Similarly, a *left Bol loop* satisfies the identity $a\cdot(x\cdot(a\cdot y))=(a\cdot(x\cdot a))\cdot y$.

As in the case of Lie groups where the tangent space at each point is equipped with Lie algebra structure, the tangent space at each point of Bol loop is equipped with the structure of Bol algebra.

Definition 2.1. *A vector space $\mathfrak{B}$ over a field K is called Bol algebra if it is equipped with a trilinear operation $[-;-,-]$ and a skew-symmetric operation $x\cdot y$ satisfying the following identities:*

(i) $[x;x,y]=0$

(ii) $[x;y,z]+[z;x,y]+[y;z,x]=0.$

(iii) $[[x;y,z];\alpha,\beta]=[[x;\alpha,\beta];y,z]+[x;[y;\alpha,\beta],z]+[x;y,[z;\alpha,\beta]]$

(iv) $[x\cdot y;\alpha,\beta]=[x;\alpha,\beta]\cdot y+x\cdot[y;\alpha,\beta]+[\alpha\cdot\beta;x,y]+[x\cdot y]\cdot[\alpha\cdot\beta]$

for all x, y, z, α and β in $\mathfrak{B}$.

In other words, a Bol algebra is a Lie triple system $(\mathfrak{B},[-;-,-])$ with an additional bilinear skew-symmetric operation $x\cdot y$ such that, the derivation $D_{\alpha,\beta}:x\to[x;\alpha,\beta]$ on a ternary operation is a pseudo-differentiation with components α, β on a binary operation, that is; for all x, y and z in $\mathfrak{B}$, we have

$$D_{\alpha,\beta}(x\cdot y)=(D_{\alpha,\beta}(x))\cdot y+x\cdot(D_{\alpha,\beta}(y))+[\alpha\cdot\beta;x,y]+(x\cdot y)\cdot(\alpha\cdot\beta).$$

$D_{\alpha,\beta}$ is a differentiation on ternary operation $[-;-,-]$ that is;

$$D_{\alpha,\beta}[x;y,w]=[D_{\alpha,\beta}(x);y,w]+[x;D_{\alpha,\beta}(y),w]+[x;y,D_{\alpha,\beta}(w)].$$

In fact, the Bol algebra defined above is called right Bol algebra. In particular, any Lie triple system may be regarded as Bol algebra with the trivial multiplication $x\cdot y=0$, for all $x,y\in\mathfrak{B}$.

Bol algebras can be realized as the tangent algebras of Bol loops with the right Bol identity, and they allow embedding in Lie algebras [6,15].

Definition 2.2. *A linear map $\varphi:\mathfrak{B}_1\to\mathfrak{B}_2$ between two Bol algebras is called morphism of Bol algebras if it is preserve the ternary and the binary operations.*

The subspace S of Bol algebra $\mathfrak{B}$ is a sub-Bol algebra if the inclusion $j:S\hookrightarrow\mathfrak{B}$ is a morphism of Bol algebras.

Definition 2.3. *Let $(\mathfrak{B},[-;-,-],\cdot)$ be a Bol algebra over a field K, a pseudo-differentiation is a linear map $P:\mathfrak{B}\to\mathfrak{B}$ for which, there exists z ∈ B (a companion of D) such that $P(x\cdot y)=P(x)\cdot y+x\cdot P(y)+[z;x,y]+(x\cdot y)\cdot z$; the companion is not necessarily unique.*

The set of all companions of D is denoted $Com(D)$. The map $D_{\alpha,\beta}:x\to[x;\alpha,\beta]$ is a pseudo-differentiation with companion α . β, called inner pseudo-differentiation of $\mathfrak{B}$. The pseudo-differentiations of B form a Lie algebra, denoted by pder $\mathfrak{B}$ under the natural product $[P,P']=PP'-P'P$. The algebra ipder $\mathfrak{B}$ generate by $\{D_{a,b}/a,b\in\mathfrak{B}\}$ is a Lie subalgebra of pder $\mathfrak{B}$, called the Lie algebra of inner pseudo-differentiations of B. The enlarged algebra of pseudo-differentiations of $\mathfrak{B}$ is defined as $Pder\,\mathfrak{B}=\{(D,z),D\in pder\,\mathfrak{B},z\in Com(D)\}$ and the enlarged algebra of inner pseudo-differentiation is defined as $Ipder\,\mathfrak{B}=\{(D,z),D\in ipder\,\mathfrak{B},z\in Com(D)\}$.

It is showed in [4,5] that, those algebras defined below are the Lie algebras with the brackets $[P,P']=PP'-P'P$

The direct sum $L=\mathfrak{B}\oplus Ipder\,\mathfrak{B}$ is a Lie algebra with the operation $[x,y]=D_{x,y}$, $[x,D_{a,b}]=D_{a,b}(x)$, for all x, y, a, b in B. The Lie algebra $(L,[,])$ is called the standard enveloping Lie algebra of Bol algebra $\mathfrak{B}$.

The map $\delta_a:x\mapsto x\cdot a$ is a linear map of B. We denote by $\overline{\mathfrak{B}}$ the Lie algebra generate by $\{\delta_a,a\in\mathfrak{B}\}$ with brackets $[\delta_a,\delta_b]=\delta_a\delta_b-\delta_b\delta_a$. We get an other Lie algebra $\overline{L}=\overline{\mathfrak{B}}\oplus Ipder\mathfrak{B}$ which is a subalgebra of the Lie algebra generated by linear maps of $\mathfrak{B}$.

If the subspace $\mathcal{I}$ of $\mathfrak{B}$ satisfies the stronger condition $\mathcal{I}\cdot\mathfrak{B}+(\mathcal{I};\mathfrak{B},\mathfrak{B})\subset\mathcal{I}$, then $\mathcal{I}$ is an ideal of B. An ideal $\mathcal{I}$ of B automatically satisfies $(\mathfrak{B};\mathcal{I},\mathfrak{B})\subset\mathcal{I}$ and $(\mathfrak{B};\mathfrak{B},\mathcal{I})\subset\mathcal{I}$.

For more understanding of Bol algebras and Bol loops, it is important to investigate about their representations. We defined a representation of Bol algebra as follows.

Definition 2.4. *If $\mathfrak{B}$ is a Bol algebra over a field K and V a vector field over K, the pair (ρ,δ) with the skew-symmetric bilinear map $\rho:\mathfrak{B}^2\to EndV$ and the linear map $\delta:\mathfrak{B}\to EndV$ is said to be a representation of Bol algebra $\mathfrak{B}$ in V if there exists a bilinear operation $\Delta:\mathfrak{B}^2\to EndV$ such that the following statements are satisfied:*

(R1) $\rho(u,v)=\Delta(u,v)-\Delta(v,u)$

(R2) $[\rho(a,b),\rho(u,v)]=\rho([a,u,v],b)+\rho(a,[b,u,v])$

(R3) $[\rho(u,v),\delta(a)]=\delta([a,u,v])+\Delta(u\cdot v,a)+\delta(u\cdot v)\delta(a)$

for all x, y, a, b in $\mathfrak{B}$.

The operation Δ is called a companion of the representation (ρ,δ) of the Bol algebra $\mathfrak{B}$.

In this case we can denoted by (ρ,δ,Δ,V) or simply (ρ,δ,Δ), the representation (ρ,δ,V) with companion Δ. Following the approach of Consuelo and Zelmanov for the representations of Jordan Superalgebras [2], it is equivalent to say that the vector space V is a Bol module ($\mathfrak{B}$-module) i.e., $E_V=\mathfrak{B}\oplus V$ possesses the structure of Bol algebra such that:

(a) $\mathfrak{B}$ is a sub-Bol algebra of E_V,

Citation: Ndoune N, Bouetou TB (2015) On Representations of Bol Algebras. J Generalized Lie Theory Appl S2: 005. doi:10.4172/1736-4337.S2-005

(b) V is an ideal of Bol algebra E_V and

(c) $x \cdot y = 0$ if both $x, y \in V$ and $[x, y, z] = 0$ if any two of x, y, z lie in V.

A particular instance where $V = B$ and we set $D(u,v) = D_{u,v}, \delta(u) = \delta_u$ the pair (D,δ) is a representation of Bol algebra with companion $\Delta(u,v) = [u,-,v]$ called *regularrepresentation* of B.

Example 2.1. *Let* $(\mathfrak{B},[-;-,-],\cdot)$ *be the Bol algebra with basis* (e_1, e_2) *over a field of complex numbers, were* $[e_1,e_2,e_1] = e_1$, $[e_2,e_1,e_2] = e_2$ *and* $e_1 \cdot e_2 = e_2$. *We recall that* $det(u,v)$ *is the determinant of the pair of vectors* (u,v) *with* $u = u_1e_1 + u_2e_2$ *and* $u = u_1e_1 + u_2e_2$. *Note that this Bol algebra arise from the classification of two-dimensional Bol algebras obtained by Kuz'min and Zaidi* [4]. *We set*

$$D(u,v) = \begin{pmatrix} -det(u,v) & 0 \\ 0 & det(u,v) \end{pmatrix}$$

$$\delta(u) = \begin{pmatrix} 0 & 0 \\ u_2 & -u_1 \end{pmatrix}$$

$$\Delta(u,v) = \begin{pmatrix} -u_1v_2 & u_1v_1 \\ u_2v_2 & -u_2v_1 \end{pmatrix}.$$

It is clear that (D,δ,Δ) is a regular representation of $\mathfrak{B}$.

Now let (ρ,δ,Δ) and (ρ',δ',Δ') be two representations of Bol algebra $\mathfrak{B}$ on V. a morphism of the representation (ρ,δ,Δ) to a representation (ρ',δ',Δ') is a linear map $f: V \to V$ such that $\rho' = f\rho$, $\delta' = f\delta$ and $\Delta' = f\Delta$. Clearly the composition of morphisms of representations is a morphism of representations. The collection of all representations and their morphisms forms a K-linear category denoted by $Rep(\mathfrak{B})$ and called the category of representations of Bol algebra $\mathfrak{B}$.

We consider $Z_1(\mathfrak{B}) = \bigcap_{y\in\mathfrak{B}} ker(-\cdot y)$ and $Z_2(\mathfrak{B}) = \bigcap_{y,z\in\mathfrak{B}} ker[-;y,z]$, the center of Bol algebra is $Z(\mathfrak{B}) = Z_1(\mathfrak{B}) \cap Z_2(\mathfrak{B})$. It is simple to see that, the kernel of the operation $<\rho,\delta>$ given by $Ker <\rho,\delta> = \{x \in \mathfrak{B} / \rho(x,\mathfrak{B}) + \delta(x) = 0\}$ is the center of $\mathfrak{B}$.

Engel's Theorem for Bol Algebras

Before giving the Engel's theorem, we first need to define and characterize the nilpotent representations.

A representation (ρ,δ,Δ) of Bol algebra $\mathfrak{B}$ in V is nilpotent if for all $x,y,z \in \mathfrak{B}$, $\rho(x,y), \delta(x)$ and $\Delta(x,y)$ are nilpotent endomorphisms; that is if there is a positive integer n such that $(\rho,\delta,\Delta)^n = 0$. Let (ρ,δ,Δ) be a representation of $\mathfrak{B}$ in V. we define the triplet $(ad_\rho, ad_\delta, ad_\Delta)$ as follows: $ad_\rho(x,y) = [\rho(x,y),-]$, $ad_\delta(x,y) = [\delta(x),-]$ and $ad_\Delta(x,y) = [\Delta(x,y),-]$.

Proposition 3.1. *With the above notations, the pair* (ad_ρ, ad_δ) *is a representation of Bol algebra* $\mathfrak{B}$ *in a vector space V with companion* ad_Δ.

Proof. The objective is to show that (R_1), (R_2) and (R_3) are satisfied. Let $a,b,u,v \in \mathfrak{B}$ and $f \in End V$. We have

$$\begin{aligned}
[ad_\rho(a,b), ad_\rho(u,v)](f) &= [ad_\rho(a,b),[ad_\rho(u,v),f]] \\
&= [\rho(a,b),[\rho(u,v),f]] - [\rho(u,v),[\rho(a,b),f]] \\
&= [[\rho(a,b),\rho(u,v)],f] \\
&= [\rho(a,b),\rho(u,v)]f - f[\rho(a,b),\rho(u,v)] \\
&= \rho([a;u,v],b)f + \rho(a,[b,u,v])f - f\rho([a;u,v],b) - f\rho(a,[b,u,v]) \\
&= (ad_\rho([a;u,v],b) + ad_\rho(a,[b,u,v]))(f)
\end{aligned}$$

Then (R_2) holds. In other hand we have

$$[ad_\rho(a,b), ad_\rho(u,v)] = ad_\rho([a;u,v],b) + ad_\rho(a,[b,u,v])$$

$$\begin{aligned}
lclad_\rho(a,b)(f) &= [\rho(a,b),f] = \rho(a,b)f - f\rho(a,b) \\
&= \Delta(a,b)f - \Delta(a,b)f - \Delta(b,a)f + f\Delta(b,a) \\
&= [\Delta(a,b),f] - [\Delta(b,a),f] \\
&= ((ad_\Delta(a,b) - ad_\Delta(b,a))(f)
\end{aligned}$$

Therefore we have the desire equality $ad_\rho(a,b) = ad_\Delta(a,b) - ad_\Delta(b,a)$. This shows that (R_1) is satisfied. Finally, we have for all $f \in End V$,

Thus $[ad_\rho(a,b), ad_\delta(u)] = ad_\delta([u;a,b]) + ad_\delta(a\cdot b)ad_\delta(u) + ad_\Delta(a\cdot b,u)$ and the desire conclusion follows, that is (R_3) is verified.

Definition 3.1. *The representation* $(ad_\rho, ad_\delta, ad_\Delta)$ *is called the adjoint representation of* (ρ,δ,Δ).

Now we give the link between nilpotent representation and adjoint representation. The above result arises to the representations of Lie algebras.

lemma 3.1. *Let* (ρ,δ,Δ) *be a representation of Bol algebra on the vector space V. If* (ρ,δ,Δ) *is nilpotent, then its adjoint representation is also nilpotent.*

Proof. Let (ρ,δ,Δ) be a nilpotent representation of Bol algebra, and $(ad_\rho, ad_\delta, ad_\Delta)$ its adjoint representation. Then there exists a positive integer p such that $(\rho)^p = 0$, $(\delta)^p = 0$ and $(\Delta)^p = 0$. If σ is one of the map ρ, δ, or Δ it is clear that $ad_\sigma = l_\sigma + h_\sigma$ where l_σ and h_σ are nilpotent. we have $(ad_\sigma)^{2p-1} = (l_\sigma + h_\sigma)^{2p-1} = 0$. Hence the result.

Now we are in position to prove our first main theorem.

Theorem 3.1. *Let* $\mathfrak{B}$ *be a finite dimensional Bol algebra over a field K and* $\mathcal{R}$ *consists of nilpotent representations of Bol algebra* $\mathfrak{B}$ *in a finite dimensional space V. Then there exists a vector* $v \in V^3$, $v \neq 0$ *such that* $(\rho,\delta,\Delta)(v) = 0$ *for all* $(\rho,\delta,\Delta) \in \mathcal{R}$.

Proof. We agree that $(\rho,\delta,\Delta)(v) = (\rho(v_1), \delta(v_2), \Delta(_3))$, where $v = (v_1, v_2, v_3)$, that is we identify (ρ,δ,Δ) by $(\rho(a,b), \delta(a), \Delta(a,b))$ for all a, b in $\mathfrak{B}$. It is clear that $\mathcal{R}$ is a subspace of $(Env)^3$ and we can define on it the following bracket $[(f,g,h),(f',g',h')] = ([f,f'],[g,g'],[h,h'])$. $(\mathcal{R},[-,-])$ is a Lie algebra.

The proof of the theorem goes by induction on $dim\mathcal{R}$. When $dim\mathcal{R} = 1$, since $\mathcal{R}$ is generated by a single nilpotent representation then the claim is immediate.

Suppose now that the claim is true for all subalgebras of nilpotent representations spaces of dimension less than $dim\mathcal{R} \geq 1$.

Since, $dim\mathcal{R} \geq 1$, we have a proper Lie subalgebra $L \subseteq \mathcal{R}$. We can choose L to be a maximal subalgebra. We show before continuing that, L has a codimension one in $\mathcal{R}$ and L is an ideal.

L acts via the adjoint operator on $\mathcal{R}$ and L. In the latter case, since $dimL < dim\mathcal{R}$, we know by Engel's theorem apply for L, that there exists a nonzero element $\overline{r} \in \mathcal{R}/L$ such that $[l,\overline{r}] = 0$ $\overline{(\rho,\delta,\Delta)} \in \mathcal{R}/L$ and $[(l_1,l_2,l_3),\overline{(\rho,\delta,\Delta)}] = \overline{0}$ for $(l_1,l_2,l_3) \in L$. We know that $\overline{(\rho,\delta,\Delta)} = (\rho,\delta,\Delta) + L$; then $(\rho,\delta,\Delta) \in \mathcal{R} - L$. It follows that $[K(\rho,\delta,\Delta) + L, L] \subseteq L$. Moreover $[K(\rho,\delta,\Delta) + L, K(\rho,\delta,\Delta) + L] \subseteq L$. These imply that $K(\rho,\delta,\Delta) + L$ is a Lie subalgebra of $\mathcal{R}$, and contains L as an ideal. By maximality of L, it follows that $Kr + L = \mathcal{R}$, so we are done.

Now we define the vector space $\mathbf{W} = \{w \in V^3 / Lw = 0\}$. Let $w = (w_1, w_2, w_3) \in \mathbf{W}$ and $(\rho,\delta,\Delta) \in L$, then $(l_1,l_2,l_3)(\rho,\delta,\Delta)(w) = 0$ for all $(l_1,l_2,l_3) \in L$. Other we have

$$\begin{aligned}
(l_1,l_2,l_3)(\rho,\delta,\Delta)(w) &= (\rho,\delta,\Delta)(l_1,l_2,l_3)(w) + [(l_1,l_2,l_3),(\rho,\delta,\Delta)](w) \\
&= [(l_1,l_2,l_3),(\rho,\delta,\Delta)](w)
\end{aligned}$$

and $[(l_1,l_2,l_3),(\rho,\delta,\Delta)]\in L$. Since L is an ideal, we have also $[(l_1,l_2,l_3),(\rho,\delta,\Delta)](w)=0$.

Now we have $\mathcal{R}=K(\rho,\delta,\Delta)+L$ for some $(\rho,\delta,\Delta)\in L$. We know that (ρ,δ,Δ) is a nilpotent operator on **W,** so $ker(\rho,\delta,\Delta)\cap\mathbf{W}\neq 0$. Let $v=(v_1,v_2,v_3)\in ker(\rho,\delta,\Delta)\cap\mathbf{W}$ such that $v\neq 0$; then any element of L and r annihilates v.

An Extension of Ado-Iwasawa Theorem to Bol Algebras

Let L be a finite-dimensional Lie algebra over a field K. The classical Ado-Iwasawa theorem asserts the existence of a finite-dimensional L-module which gives a faithful representation of L. However, Filippov proved [17] showed that this theorem does not hold for Malcev algebras, that is homogeneous Bol algebras. Thus it is not hold for general Bol algebras.

For the Lie algeras, the Poincaré-Birkhoff-Witt theorem says that any Lie algebra L is a subalgebra of A^- for some unital associative algebra A. In the case that L is finite dimensional, the Ado-Iwasawa theorem says that A can be taken finite dimensional too. This extension of Ado-Iwasawa theorem was established for the Malcev algebras by Pérez-Izqquierdo and Shestakov [14]. There is a version of the Poincaré-Birkhoff-Witt theorem for Bol algebra proved by Kuz'min and Zaidi [4]. Now let $\mathfrak{B}$ be a Bol algebra [14] that there is an alternative algebra A and an injective map $\mathfrak{B}\to RN_{alt}(A)$, where $RN_{alt}(A)=\{a\in A/(x,a,y)=-(x,y,a)\}$ is the generalized right alternative nucleus. In this section we prove that if $\mathfrak{B}$ is a finite-dimensional Bol algebra then A can be taken finite dimension too. Our second main result is the following.

Theorem 4.1. *Let $\mathfrak{B}$ be a finite-dimensional right Bol algebra over a field of characteristic $\neq 2,3$. Then there exists a unital finite-dimensional algebra A and a monomorphism of Bol algebra $j:\mathfrak{B}\to RN_{alt}(A)$.*

Proof. Let $\mathfrak{B}$ be a Bol algebra, according to Pérez-Izquierdo [5], there exists a linear map $j:\mathfrak{B}\to RN_{alt}(U(\mathfrak{B}))$, $a\mapsto a$ such that $j(a\cdot b)=ab-ba$ and $j(a,b,c)=(ab)c-(ac)b-[b,c]a$, where $U(\mathfrak{B})$ is the universal enveloping algebra of $\mathfrak{B}$. Since $RN_{alt}(U(\mathfrak{B}))$ is closed under the binary product $[-,-]$ given by the commutators and the ternary operation $[a,b,c]=(ab)c-(ac)b-[b,c]a$ for all a,b,c in $RN_{alt}(U(\mathfrak{B}))$. By the methods of Pérez-Izquierdo [5], $RN_{alt}(U(\mathfrak{B}))$ with the binary and ternary operations defined above has the structure of Bol algebra. Thus j is a monomorphism of Bol algebras. Let $E_{\mathfrak{B}}$ be the Lie enveloping algebra of $\mathfrak{B}$. Then $E_{\mathfrak{B}}=E_+\oplus E_-$ is the $\mathbb{Z}_2$-gradation and $E_-\approxeq\mathfrak{B}$ as vector space. According to Pérez-Izquierdo and Shestakov [14], there exists a two side ideal $\mathcal{I}\subseteq U(\mathfrak{B})$ of finite codimension. Then $A=U(\mathfrak{B})/\mathcal{I}$ is a unital finite-dimensional algebra and there exists an injective map $j:\mathfrak{B}\to U(\mathfrak{B})$. The injective map j induces a monomorphism of Bol algebras $j:\mathfrak{B}\to RN_{alt}(A)$.

The Category of Representations of Bol Algebra

We give a relation between the category of representation of Bol algebra $\mathfrak{B}$ and the category of representations of its universal enveloping algebra. As immediate consequence, we show that the representation category of a Bol algebra is monoidal, or tensor category. We recall that the category of representations of Bol algebras is $Rep(\mathfrak{B})$, and the one of finite dimensional representations of Bol algebra is $rep(\mathfrak{B})$. Let $A=(A,\cdot,\Delta,\varepsilon)$ be a bialgebra, Mod(A) means the category of left A-modules (ie., representations of A). If U, V are left A-modules, then the tensor product becomes a left A-module with multiplication rule $a\cdot(u\otimes v)=\Delta(a)\cdot(u\otimes v)$ for all $a\in A$, $u\in U$ and $v\in V$. The field K is also a left A-module by $a\cdot\varsigma=\varepsilon(a)\varsigma$. The category of left A-modules is equivalent to the category of (A, A)-bimodules. Any (A, A)-bimodule can be considered as left module over $A\otimes A^{op}$, where A^{op} is define on the same space as A, by new multiplication $x\cdot y=y\cdot x$. We know in virtue of Pérez-Izquierdo [5] that for a given Bol algebra $(\mathfrak{B},[-,-],[-,-,-])$ there exists a universal enveloping $U(\mathfrak{B})$ endowed with the structure of bialgebra, that is $(U(\mathfrak{B}),\cdot,\Delta,\varepsilon)$ is a bialgebra. Analogously we denote $Rep(U(\mathfrak{B}))$ the category of representation of the bialgebra $(U(\mathfrak{B}),\cdot,\Delta,\varepsilon)$. Now we state an equivalent characterization of the representation category $Rep(\mathfrak{B})$. We prove our third main result.

Theorem 5.1. *The category of representations of Bol algebra $Rep(\mathfrak{B})$ is equivalent to the category of representations of its universal enveloping algebra $Rep(U(\mathfrak{B}))$.*

Proof. We recall that $Rep(\mathfrak{B})$ is the category of modules over the Bol algebra $\mathfrak{B}$. Following the consideration of Consuelo and Zelmanov [13], apply for the modules over Bol algebras, every B-module has the form $E_V=\mathfrak{B}\bigoplus V$, where V is a vector space over a field K and E_V possesses the structure of Bol algebra such that:

(a) $\mathfrak{B}$ is a sub-Bol algebra of E_V,

(b) V is an ideal of Bol algebra E_V and

(c) $x\,.\,y$=0 if both $x, y\in V$ and $[x, y, z] = 0$ if any two of x, y, z lie in V.

We define the multiplication $U(\mathfrak{B})\times V\to V$ by $a\cdot x=\varepsilon(a)\cdot x$. We consider the following mapping defined from $Rep(\mathfrak{B})$ to Mod($U(\mathfrak{B})$) define on the objets by $F(E_V)=V$. The map F is naturally extended on the morphisms. If U and V are the images of E_U and E_V under F, in virtue of Pérez-Izqquierdo [5] there exits a map $\mu:\mathfrak{B}\to U(\mathfrak{B})\otimes U(\mathfrak{B})$ with $\mu(a)=a\otimes 1+1\otimes a$. This implies that $U\otimes V$ is a $(U(\mathfrak{B})$-module.

Conversely, let V be a $(U(\mathfrak{B})$-module, in virtue of Pérez-Izquierdo [5] there exist an injective map $\eta:\mathfrak{B}\to U(\mathfrak{B})$. We define the multiplication $\mathfrak{B}\times V\to V$ by $a\cdot x=\eta(a)\cdot x$. Then V has the structure of module. We set now the mapping G from Mod($U(\mathfrak{B})$ to $Rep(\mathfrak{B})$ by $G(V)=E_V$. It remains to define the image of $U\otimes V$. Let E_U and E_V be two modules over $\mathfrak{B}$, We set $E=\mathfrak{B}\oplus U\otimes V$. We define the binary operation by $[a,u\otimes v]_{\otimes}=[a,u]\otimes v$; $[a,u\otimes v]_{\otimes}=[a,u]\otimes v$ and a ternary by $[a,b,u\otimes v]_{\otimes}=[a,b,u]\otimes v$; $[a,u\otimes v,b]_{\otimes}=[a,u,b]\otimes v$ and $[a,b,u\otimes v]_{\otimes}=[a,b,u]\otimes v$ for all a, b in $\mathfrak{B}$, u in V and v in V. We assume also that the restrictions of $[-,-]_{\otimes}$ and $[-,-,-]_{\otimes}$ on $\mathfrak{B}$ correspond respectively to the binary and ternary operations of B; and $x\cdot y=0$ if both $x,y\in U\otimes V$ and $[x,y,z]=0$ if any two of x,y,z lie in $U\otimes V$.

It remains to show that $(E,[-,-]_{\otimes},[-,-,-]_{\otimes})$ is a Bol algebra, that is the conditions (i) - (iv) hold. By the definition, the condition (i) is satisfied. Now let x,y,z,α,β in $\mathfrak{B}$; u in U and v in V. We have

$$\begin{aligned}[x;y,u\otimes v]+[u\otimes v;x,y]+[y;u\otimes v,x] &= [x;y,u]\otimes v+[u;x,y]\otimes v+[y;z,u]\otimes v\\ &= ([x;y,u]+[u;x,y]+[y;z,u])\otimes v\\ &= 0,\end{aligned}$$

this shows that (ii) is true.

Now let us show that (iii) holds. We have

$$\begin{aligned}[[x;y,u\otimes v];\alpha,\beta] &= [[x;y,u]\otimes v;\alpha,\beta]\\ &= [[x;y,u];\alpha,\beta]\otimes v\\ &= ([[x;\alpha,\beta];y,u]+[x;[y;\alpha,\beta],u]+[x;y,[u;\alpha,\beta]])\otimes v\\ &= [[x;\alpha,\beta];y,u\otimes v]+[x;[y;\alpha,\beta],u\otimes v]+[x;y,[u\otimes v;\alpha,\beta]]\end{aligned}$$

One can show that the above equality holds for any x,y,α,β stands for $u\otimes v$. That is (iii) holds.

Finally, we have

$$\begin{aligned}[][u\otimes v;y];\alpha,\beta] &= [[u;y]\otimes v;\alpha,\beta]\\ &= [[u;y];\alpha,\beta]\otimes v\\ &= ([u;\alpha,\beta]\cdot y+[u,[y;\alpha,\beta]]+[[\alpha,\beta];u,y]+[[u,y],[\alpha,\beta]])\otimes v.\end{aligned}$$

Thus $[[u\otimes v;y];\alpha,\beta]=[u\otimes v;\alpha,\beta]\cdot y+[u\otimes v,[y;\alpha,\beta]]+[[\alpha,\beta];u\otimes v,y]+[[u\otimes v,y],[\alpha,\beta]]$. One can show this equality for any y, α, β stands for $u\otimes v$. This completes the proof.

Definition 5.1. *A monoidal (tensor) category* $(\mathcal{C},\otimes,\mathbf{1},\alpha,\lambda)$ *is a category* $\mathcal{C}$ *equipped with tensor functor* $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ *, with a fix objet* $\mathbf{1}$ *(called the unit of a tensor category),* $\alpha:\otimes\circ(\otimes\times Id)\to\otimes\circ(Id\times\otimes)$, $\lambda:\mathbf{1}\otimes-\to Id$, $-\otimes\mathbf{1}\to Id$ *are natural isomorphisms such that the associativity and unitary constraints hold, or equivalently the pentagon and the triangle diagrams are commutative* [18-20].

We can now give a special characterization of the category of representations of Bol algebra as a consequence of the above proposition.

Corollary 5.1. *Every category of representations of Bol algebras is a monoidal category.*

Proof. It was proved by Kassel [20] that $(A,\cdot,\Delta,\varepsilon)$ is bialgebra if and only if the category Mod(A) is monoidal category. In virtue of Theorem 5.0.6, the category of representations of Bol algebra is equivalent to the category of representations of its enveloping algebra endowed with bialgebra structure. Hence the category *Rep*(B) is monoidal.

More recently it was proved by Huang and Torecillas [21], that the path coalgebra *KQ* of a given quiver *Q* always admits a bialgebra structure. So the monoidal category arising from this quiver bialgebra is the category of representations of the bialgebra *KQ*. This leads to the following conjecture.

Conjecture 5.1. *Find necessary and sufficient conditions for the existence of quiver Q such that the monoidal category arising from quiver bialgebra KQ is the category of representations of a Bol algebra over algebraically closed field K.*

A monoidal category is said to be finite, if it is equivalent to the category of finite dimensional comodules over the finite dimensional coalgebra. Thus the category *Rep*(B) of finite dimensional representations is finite monoidal category. This is a particular case of tensor categories of Etingof et al. [19]. The particular case where *Q* is a quiver without loops and 2-cyles should leads to strong relation between Bol algebras and *cluster algebras* of Fomin and Zelevinsky [22,23] for more details. In the same vein, it has been shown in literature of Schauenburg [24] that if *A* is a finite dimensional bialgebra, then *A* is Hopf algebra if and only if the category of finitely generated *A*-modules is rigid, that is finitely generate modules admit dual objets. This allows us to the following conjecture.

Conjecture 5.2. *Find necessary and sufficient conditions for a finite dimensional Bol algebra to have Hopf algebra as universal enveloping algebra.*

Representations of Free Bol Algebra *Bol*[*X*] of Finite Dimension

Let $X=\{x_1,x_2,...,x_n\}$, we construct the set of binary-ternary monomials $BT[X]$, and we assume that $BT[X]$ is closed under $[-,-]$ and $[-,-,-]$. Let $BT[X]=\{\sum_{=1}^{n}\alpha_i x_i \mid \alpha_i\in\ \}$ be the space spanned by *X*. We define the multiplication by the following rules: if $f=\sum_{i=1}^{n}\alpha_i x_i$, $g=\sum_{j=1}^{n}\beta_j x_j$ and $h=\sum_{k=1}^{n}\gamma_k x_k$ in $BT[X]$, then $[f,g]=\sum_{i,j=1}^{n}\alpha_i\beta_j[x_i,x_j]$, $[f,g,h]=\sum_{i,j,k=1}^{n}\alpha_i\beta_j\gamma_k[x_i,x_j,x_k]$. The free Bol algebra *Bol*[*X*] is the free binary-ternary algebra $BT[X]$ satisfying the identities (*i*) - (*iv*). The Bol types of degree *m* are always to construct a product of degree *m* in *Bol*[*X*]. For general construction and more details of the free Bol algebra *Bol*[*X*] [25,26]. In studies of Peresi [26] it has been shown that any multilinear identity *f* of degree *m* can be written as a linear combination of multilinear monomials. We denote the Bol types of degree *m* by B_1, B_2, …, $B_{b(m)}$, that is $f=f_1+..+f_{b(m)}$, where f_k is a linear combination of polynomial having Bol type *k*. Therefore the author regards *f* as an element of $b(m)$ copies of $\mathbb{F}S_m$, where $\mathbb{F}S_m$ is group algebra of the group of permutation S_m. Applying the representation $\Phi_\sigma:\mathbb{F}S_m\to Md_\sigma(\mathbb{F})$, ($\sigma$ partition of *m*) of S_m to *f* we obtain the representation matrix of f in partition σ: $(\Phi_\sigma(f_1)\mid\Phi_\sigma(f_2)\mid...\mid\Phi_\sigma(f_{b(m)}))$.

Now let *V* be finite dimensional space, dim(V) = *s* and $\mathfrak{B}$ is a Bol algebras of dimension *n*. Give a representation (ρ,δ,Δ) of $\mathfrak{B}$ over the space *V* is equivalent to give the matrix $(D(u,v)\mid\delta(u)\mid\Delta(u,v))$, where $D(u,v)$, $\Delta(u,v)$ are $s\times n$ $s\times n$ matrices and $\delta(u)$ is also a $s\times n$ matrix. Hence the block matrix $(D(u,v)\mid\delta(u)\mid\Delta(u,v))$ is a $(3n)\times s$ matrix.

In the special case where $\mathfrak{B}=Bol[X]$, $K=\mathbb{F}$ and V S, with Bol types B_1, B_2, …, $B_{b(m)}$ the representation matrix $(\Phi_\sigma(f_1)\mid\Phi_\sigma(f_2)\mid...\mid\Phi_\sigma(f_{b(m)}))$ of *f* corresponds to the matrix δ_f, that is the expression $\delta(f)=(\delta(f_1)\mid\delta(f_2)\mid...\mid\delta(f_{b(m)}))$. At this specific case mentioned by Peresi and Jacobson [26,27], the representation of element *f* is understood as a the representation of Bol algebra *Bol*[*X*] given by the matrix $(D(f,0)\mid\delta(f)\mid\Delta(f,0))$.

Actually we recall the classification theorem of Kuz'min and Zaidi for two-dimensional Bol algebras [4] which states as follows.

Theorem 5.2. (Kuz'min-Zaidi). *Every Bol algebra* B *of dimension two over* $\mathcal{R}$ *has a canonical basis* (e_1,e_2) *in which its multiplication table is one of the following:*

I. $[e_1,e_2]=0$, $[e_2,e_1,e_2]=\varepsilon_1 e_1$, $[e_1,e_2,e_1]=\varepsilon_2 e_2$, where $(\varepsilon_1,\varepsilon_2)=(0,0)$, $(-1,0)$, $(1,0)$, $(1,-1)$, $(1,1)$, $(-1,-1)$

II. $[e_1,e_2]=e_2$, $[e_2,e_1,e_2]=\varepsilon e_1$, $[e_1,e_2,e_1]=\beta e_2$, where $\varepsilon=0,-1,1$; $[e_2,e_1,e_2]=e_2$, $[e_1,e_2,e_1]=e_1$.

Now we are in position to prove our classification result for regular representations of the two-dimensional Bol algebras.

Theorem 5.3. *Every regular representation of two-dimensional Bol algebra* B *over K is up to equivalence of matrices given by one of the following matrices:*

$$(i)\quad R_1(u,v)=\begin{pmatrix}0 & \varepsilon_1 det(u,v) & 0 & 0 & u_2v_2\varepsilon_1 & -u_2v_1\varepsilon_1\\ -\varepsilon_2 det(u,v) & 0 & 0 & 0 & -u_1v_2\varepsilon_2 & u_1v_1\varepsilon_2\end{pmatrix}$$

$$(ii)\quad R_2(u,v)=\begin{pmatrix}0 & \varepsilon det(u,v) & 0 & 0 & u_2v_2\varepsilon & -u_2v_1\varepsilon\\ -\beta det(u,v) & 0 & u_2 & -v_1 & -u_1v_2\beta & u_1v_1\beta\end{pmatrix}$$

$$(iii)\quad R_3(u,v)=\begin{pmatrix}-det(u,v) & 0 & 0 & 0 & u_1v_2 & u_1v_1\\ 0 & det(u,v) & u_2 & -v_1 & u_2v_2 & -u_2v_1\end{pmatrix}$$

Proof. In virtue of classification theorem of Kuz'min and Zaidi [4], every Bol algebra of dimension two is of type (*I*) or of type (*II*) by using the items of their theorem.

We suppose in the first case that our Bol algebra is of type (*I*), that is B has a canonical basis *(e_1,e_2)* in which its multiplication table is given by $[e_1,e_2]=0$, $[e_2,e_1,e_2]=\varepsilon_1 e_1$, $[e_1,e_2,e_1]=\varepsilon_2 e_2$, where

$(\varepsilon_1,\varepsilon_2)=(0,0)$, $(-1,0)$, $(1,0)$, $(1,-1)$, $(1,1)$, $(-1,-1)$.

Let u and v be the two vectors of B, with $u=u_1e_1+u_2e_2$ and $u=u_1e_1+u_2e_2$. We have $D(u,v)(e_1)=u_1v_2[e_1,e_1,e_2]+u_2v_1[e_1,e_2,e_1]$. Since $[e_1,e_1,e_2]=-[e_1,e_2,e_1]$, we have

$$\begin{aligned} D(u,v)(e_1) &= -u_1v_2[e_1,e_2,e_1]+u_2v_1[e_1,e_2,e_1] \\ &= (-u_1v_2+u_2v_1)\varepsilon_2 e_2 \\ &= -det(u,v)\varepsilon_2 e_2, \end{aligned}$$

We have also

$$\begin{aligned} D(u,v)(e_2) &= u_1v_2[e_2,e_1,e_2]+u_2v_1[e_2,e_2,e_1] \\ &= (u_1v_2-u_2v_1)\varepsilon_1 e_1 \\ &= det(u,v)\varepsilon_1 e_1. \end{aligned}$$

Thus $D(u,v)=\begin{pmatrix} 0 & \varepsilon_1 det(u,v) \\ -\varepsilon_2 det(u,v) & 0 \end{pmatrix}$.

Now we compute the matrix of $\Delta(u,v)$ as follows. We have

$$\begin{aligned} \Delta(u,v)(e_1) &= u_1v_2[e_1,e_1,e_2]+u_2v_2[e_2,e_1,e_2] \\ &= u_2v_2\varepsilon_1 e_1 - u_1v_2\varepsilon_2 e_2, \end{aligned}$$

and

$$\begin{aligned} \Delta(u,v)(e_2) &= u_1v_1[e_1,e_2,e_1]+u_2v_1[e_2,e_2,e_1] \\ &= -u_2v_1\varepsilon_1 e_1 + u_1v_1\varepsilon_2 e_2, \end{aligned}$$

hence $\Delta(u,v)=\begin{pmatrix} u_2v_2\varepsilon_1 & -u_2v_1\varepsilon_1 \\ -u_1v_2\varepsilon_2 & u_1v_1\varepsilon_2 \end{pmatrix}$. Because $[e_1,e_2]=0,$ we have $\delta(u)=0.$ Therefore the bloc matrix $(D(u,v)\,|\,\delta(u)\,|\,\Delta(u,v))$ corresponds to the matrix $R_1(u,v)$.

The second case corresponds to Bol algebra of type (I), that is B has a canonical basis (e_1,e_2) in which its multiplication table is given by $[e_1,e_2]=e_2$, $[e_2,e_1,e_2]=\varepsilon e_1$, $[e_1,e_2,e_1]=\beta e_2$, where $\varepsilon=0,-1,1;$ $[e_2,e_1,e_2]=e_2$, $[e_1,e_2,e_1]=e_1$.

If $[e_1,e_2]=e_2$, $[e_2,e_1,e_2]=\varepsilon e_1$, $[e_1,e_2,e_1]=\beta e_2$, where $\varepsilon=0,-1,1;$ we use the analogous methods as at the first case to get $D(u,v)=\begin{pmatrix} 0 & \varepsilon det(u,v) \\ -\beta det(u,v) & 0, \end{pmatrix}$ $\delta(u)=\begin{pmatrix} 0 & 0 \\ u_2 & -u_1 \end{pmatrix}$ and $\Delta(u,v)=\begin{pmatrix} u_2v_2\varepsilon & -u_2v_1\varepsilon \\ -u_1v_2\beta & u_1v_1\beta \end{pmatrix}$. Hence the bloc matrix $(D(u,v)\,|\,\delta(u)\,|\,\Delta(u,v))$ corresponds to the matrix $R_2(u,v)$.

Finally, for $[e_1,e_2]=e_2$ and $[e_2,e_1,e_2]=e_2$, $[e_1,e_2,e_1]=e_1$, we have $(D(u,v)\,|\,\delta(u)\,|\,\Delta(u,v))=\begin{pmatrix} -det(u,v) & 0 & 0 & 0 & u_1v_2 & u_1v_1 \\ 0 & det(u,v) & u_2 & -v_1 & u_2v_2 & -u_2v_1 \end{pmatrix}$, this end the proof.

Acknowledgement

We aim at sending our sincere gratitude to the reviewers for their reports on our paper and their availability. The second author thanks the IHES for hospitality through the Launsbery Foundation and, the IMU for travel support throughout a grant from the Simons Foundation during the writing of this paper.

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Citation: Ndoune N, Bouetou TB (2015) On Representations of Bol Algebras. J Generalized Lie Theory Appl S2: 005. doi:10.4172/1736-4337.S2-005

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, University of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications

Merger and Borzi, J Generalized Lie Theory Appl 2016, S2
http://dx.doi.org/10.4172/1736-4337.S2-006

Research Article Open Access

A Lie Algebraic and Numerical Investigation of the Black-Scholes Equation with Heston Volatility Model

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Abstract

This work deals with an extension of the Black-Scholes model for rating options with the Heston volatility model. A Lie-algebraic analysis of this equation is applied to reduce its order and compute some of its solutions. As a result of this method, a five-parameter family of solutions is obtained. Though, these solutions do not match the terminal and boundary conditions, they can be used for the validation of numerical schemes.

Keywords: Lie algebra; Black-Scholes equation; Differential equations; Lie symmetries; Diffeomorphisms

Introduction

Black and Scholes [1] assumed a financial market, where a risk free bond with constant interest rate r, an asset with price S that is modelled by a geometric Brownian motion, and call and put options related to this asset can be traded. With the assumption of an arbitrage free market and in the framework of It $\bar{o}$'s stochastic differential equations, it is possible to derive the well-known Black-Scholes partial differential equation for the fair price of an option V. As the assumptions of this first modelling attempt are in practice too restrictive, several extensions of this model were proposed. One possible direction is to discard the assumption of a constant volatility for the geometric Brownian motion of the asset price and to assume that it is itself a random variable governed by a stochastic differential equation [2]. The resulting stochastic differential problem is given by

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \tag{1a}$$

$$dv_t = \alpha\left(m - v_t\right)dt + L\sqrt{v_t} dW_t^{(2)}, \tag{1b}$$

where we suppose that $W_t^{(1)}, W_t^{(2)}$ are two stochastically independent Wiener processes. The model constants α, m and L are supposed to be positive. The drift term of (1b) is built in a way such that the average of v_t tends to have approximately the value m. In particular, if L is zero then v_t is deterministic and converges exponentially to m as t tends to infinity. In this case, the option price behaves according to the solution of the usual Black-Scholes equation with a constant volatility $\sigma = \sqrt{m}$.

Based on (1), it is possible to derive a partial differential equation for the price of an option V [2] as it has been done for the model with constant volatility in studies of Gunther and Jungel [3]. In contrast to the standard Black-Scholes equation the PDE that arises with Heston's volatility model involves one more argument representing the current volatility of the market. The resulting two-dimensional Black-Scholes equation is as follows.

$$V_t + \frac{1}{2}x^2 y V_{xx} + \frac{1}{2}L^2 y V_{yy} + rxV_x + (\alpha m - (\alpha + \lambda)y)V_y - rV = 0. \tag{2}$$

In application, this equation is augmented with the following terminal and boundary conditions

$$\begin{aligned}
V(x,y,T) &= [a(x-K)]^+, \\
V(0,y,t) &= \frac{1-a}{2} K e^{-r(T-t)}, \\
\lim_{x\to\infty} \partial_x V(x,y,t) &= \frac{1+a}{2}, \\
\lim_{y\to\infty} V(x,y,t) &= \begin{cases} x & \text{if } a = +1 \\ K e^{-r(T-t)} & \text{if } a = -1 \end{cases}
\end{aligned} \tag{3}$$

In this setting, a denotes whether a call (a = 1) or a put (a = −1) option is considered. T represents the time when one is allowed to buy or sell a share of an asset for the prescribed price K, whereas λ is the parameter that models the price of volatility risk [2]. In (2), x represents the asset price S and y denotes the current volatility v. As the asset price and the current volatility are always positive, we are searching for a solution of the above differential problem (2)-(3) in the domain $\Omega = [0,\infty)\times[0,\infty)\times[0,T]$.

In the following, analytical and numerical solutions of (2) together with the boundary and terminal conditions (3) are sought. In particular, a quick review of Lie symmetries of partial differential equations is given in Section 2. In Section 3, this method is applied to the 2-dimensional Black-Scholes equation (2) and we derive a five-parameter family of analytical solutions. In Section 4, convergence properties of the Chang-Cooper discretization are tested with the given analytical solutions. A section of conclusion completes the exposition of our work.

Lie Theoretical Analysis of Differential Equations

In this section, we illustrate how Lie symmetries can be used to determine analytical solutions of partial differential equations. Applications of this method can be found in literature of Bordag [4] and Naicker V, Andriopoulos K, Leach [5]. Our review is based on the book of Stephani [6].

Many partial differential equations for a function u that is dependent on n variables x_i ($i = 1,\ldots,n$) can be written as follows

$$H(y_k) = 0, \tag{4}$$

with an analytic function H, where y_k denotes subsequently the independent variables x_i, the dependent variable u, and its derivatives $u_{i_1,\ldots,i_j} = \frac{\partial^j u}{\partial x_{i_1}\ldots\partial x_{i_j}}$. Equation (4) defines a manifold in some multi-

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Received December 19, 2015; **Accepted** January 25, 2016; **Published** January 27, 2016

Citation: Merger J, Borzi A (2016) A Lie Algebraic and Numerical Investigation of the Black-Scholes Equation with Heston Volatility Model. J Generalized Lie Theory Appl S2: 006. doi:10.4172/1736-4337.S2-006

Citation: Merger J, Borzi A (2016) A Lie Algebraic and Numerical Investigation of the Black-Scholes Equation with Heston Volatility Model. J Generalized Lie Theory Appl S2: 006. doi:10.4172/1736-4337.S2-006

dimensional Euclidean space and its solutions are sub-manifold. Diffeomorphisms of $\mathbb{R}^n$ can be used to permute the set of solution and find solutions with special properties. Therefore, we observe that a one parameter group of diffeomorphisms $G=\{f^\varepsilon:\mathbb{R}^n\to\mathbb{R}^n \mid \varepsilon\in\mathbb{R}\}$ can be completely determined by the first order differential operator

$$X=\sum_{i=1}^{n}\xi_i\frac{\partial}{\partial x_i},\quad \text{where } \xi_i=\frac{d}{d\varepsilon}f_i^\varepsilon(x_1,\ldots,x_n)\Big|_{\varepsilon=0}, \tag{5}$$

which is called infinitesimal generator or symbol. In fact, the group action on a point $x_0\in\mathbb{R}^n$ can be computed by solving the initial value problem

$$\frac{d}{d\varepsilon}f^\varepsilon(x_0)=\xi(f^\varepsilon(x_0)),\quad f^0(x_0)=x_0. \tag{6}$$

Moreover, the coefficients of the symbol $\widetilde{X}=\sum_{i=1}^n\eta_i\frac{\partial}{\partial y_i}$, with respect to a change of variables $y=T(x)$ with a transformation $T:\mathbb{R}^n\to\mathbb{R}^n$, can be computed as follows

$$\eta_i(y)=X(T_i)(T^{-1}(y)), \tag{7}$$

where $X(T_i)$ denotes the application of the first order differential operator X on the function T_i. It can be shown that there always exists a set of canonical variables in which the symbol has the normal form $\widetilde{X}=\frac{\partial}{\partial y_1}$; [7].

As the symbol X acts only on the independent variables, it is prolonged to act in a higher dimensional space including also the dependent variable and its derivatives up to the order of the partial differential equation. The prolonged infinitesimal generator is defined as follows

$$X=\sum_{i=1}^n\xi_i\frac{\partial}{\partial x_i}+\eta\frac{\partial}{\partial u}+\sum_{i=1}^n\eta_i\frac{\partial}{\partial u_i}+\sum_{i_1,i_2=1}^n\eta_{i_1,i_2}\frac{\partial}{\partial u_{i,j}}+\ldots;$$

where the coefficients $\eta_{i_1\ldots i_s}$ are given by

$$\eta_{i_1\ldots i_s}=\frac{D^s}{Dx_{i_1}\ldots Dx_{i_s}}\left(\eta-\sum_{j=1}^n u_j\xi_j\right)+\sum_{j=1}^n u_{i_1\ldots i_s j}\xi_j; \tag{8}$$

Here, the total differentiation operator $\frac{D}{Dx_i}=\frac{\partial}{\partial x_i}+u_i\frac{\partial}{\partial u}+\sum_{j=1}^n u_{ij}\frac{\partial}{\partial u_j}+\ldots$ is used.

A Lie symmetry of a PDE is defined as a group of transformations of the independent and dependent variables such that set of solutions is invariant under these transformations. From the fact that the image of a solution satisfies the PDE, i.e. $H(f^\varepsilon(y_k))=0$ for all ε, it can be shown that

$$X(H(y_k))=0, \tag{9}$$

holds everywhere on the solution manifold $H(y_k)=0$, where X is the prolonged symbol of the transformation group. For a given Lie symmetry, we seek to find its canonical variables w_k as the corresponding symbol is $\widetilde{X}=\frac{\partial}{\partial w_1}$ and (9) then reads as $\frac{\partial}{\partial w_1}H(w_k)=0$. Hence, the resulting PDE written in the new variable w_k is independent of w_1 and therefore involves one independent parameter less. Computing solutions of the transformed PDE, which should be easier as less independent variables are involved, and reversing the transformation, provides solutions of the original PDE.

Lie Analysis of the Two-dimensional Black-Scholes Equation

In this section, the Lie method is used to find solutions to the 2-dimensional Black-Scholes equation (2) written as $H=0$, where H is defined as follows

$$H(x,y,t,V,V_x,V_y,V_t,V_{xx},V_{yy}):=V_t+\frac{1}{2}x^2yV_{xx}+\frac{1}{2}L^2yV_{yy}+rxV_x+(\alpha m-(\alpha+\lambda)y)V_y-rV.$$

We assume that this equation admits a Lie symmetry with the infinitesimal generator

$$X=\xi(x,y,t,V)\frac{\partial}{\partial x}+\gamma(x,y,t,V)\frac{\partial}{\partial y}+\tau(x,y,t,V)\frac{\partial}{\partial t}+\varphi(x,y,t,V)\frac{\partial}{\partial V}.$$

Then, we first apply the prolonged symbol

$$X=\xi\frac{\partial}{\partial x}+\gamma\frac{\partial}{\partial y}+\tau\frac{\partial}{\partial t}+\varphi\frac{\partial}{\partial V}+\varphi_1\frac{\partial}{\partial V_x}+\varphi_2\frac{\partial}{\partial V_y}+\varphi_3\frac{\partial}{\partial V_t}+\varphi_{11}\frac{\partial}{\partial V_{xx}}+\varphi_{22}\frac{\partial}{\partial V_{yy}}$$

to the function H and then evaluate the resulting function $X(H)$ on the solution manifold $H=0$. The resulting expression yields zero, whenever X is the generator of a Lie symmetry. The exact expressions for the prolonged coefficients ϕ_1, ϕ_2, ϕ_3, ϕ_{11} and ϕ_{22} according to (8) are given by

$$\begin{aligned}
\varphi_1&=\varphi_x+\varphi_V V_x-\xi_x V_x-\gamma_x V_y-\tau_x V_t-\xi_V V_x V_x-\gamma_V V_x V_y-\tau_V V_x V_t,\\
\varphi_2&=\varphi_y+\varphi_V V_y-\xi_y V_x-\gamma_y V_y-\tau_y V_t-\xi_V V_y V_x-\gamma_V V_y V_y-\tau_V V_y V_t,\\
\varphi_3&=\varphi_t+\varphi_V V_t-\xi_t V_x-\gamma_t V_y-\tau_t V_t-\xi_V V_t V_x-\gamma_V V_t V_x-\tau_V V_t V_t,\\
\varphi_{11}&=\varphi_{xx}+2\varphi_{xV}V_x+\varphi_V V_{xx}+\varphi_{VV}V_xV_x-\xi_{xx}V_x-\gamma_{xx}V_y-\tau_{xx}V_t\\
&-2\xi_xV_{xx}-2\gamma_xV_{xy}-\tau_xV_{xt}-2\xi_{xV}V_xV_x-2\gamma_{xV}V_xV_y-2\tau_{xV}V_xV_t\\
&-\xi_VV_xV_{xx}-2\xi_VV_xV_{xx}-\gamma_VV_yV_{xx}-2\gamma_VV_xV_{xy}-\tau_VV_tV_{xx}-2\tau_VV_xV_{xt}\\
&-\xi_{VV}V_xV_xV_x-\gamma_{VV}V_xV_xV_y-\tau_{VV}V_xV_xV_t,\\
\varphi_{22}&=\varphi_{yy}+2\varphi_{yV}V_y+\varphi_VV_{yy}+\varphi_{VV}V_yV_y-\xi_{yy}V_x-\gamma_{yy}V_y-\tau_{yy}V_t\\
&-2\xi_yV_{yx}-2\gamma_yV_{yy}-\tau_yV_{yt}-2\xi_{yV}V_yV_x-2\gamma_{yV}V_yV_y-2\tau_{yV}V_yV_t\\
&-\xi_VV_xV_{yy}-2\xi_VV_yV_{yx}-\gamma_VV_yV_{yy}-2\gamma_VV_yV_{yy}-\tau_VV_tV_{yy}-2\tau_VV_yV_{yt}\\
&-\xi_{VV}V_yV_yV_x-\gamma_{VV}V_yV_yV_y-\tau_{VV}V_yV_yV_t,
\end{aligned}$$

where sub-indices x, y, and V of ξ, γ, τ and ϕ denote partial derivatives with respect to the given variables. The equation $H=0$ is solved for V_t and inserted into $X(H)=0$. Afterwards this single equation splits up into the determining equations, since the derivative variables (V_x, V_y, V_{xx}, …) are linearly independent. Among the resulting equations, there are simple ones as $\xi_V=\gamma_V=\tau_x=\tau_y=\tau_V=\varphi_{VV}=0$. Hence, we solve the following remaining system of partial differential equations

$$0=L^2\xi_y+x^2y\gamma_x \tag{I}$$

$$0=\gamma+y(\tau_t-2\gamma_y) \tag{II}$$

$$0=\xi+x(\gamma_y-\xi_x) \tag{III}$$

$$0=\xi_t+\frac{1}{2}x^2y(\xi_{xx}-2\varphi_{Vx})+\frac{1}{2}L^2y\xi_{yy}+rx(\xi_x-\tau_t)+(\alpha m-(\alpha+\lambda)y)\xi_y-r\xi \tag{IV}$$

$$0=\gamma_t+\frac{1}{2}x^2y\gamma_{xx}+\frac{1}{2}L^2y(\gamma_{yy}-2\varphi_{Vy})+rx\gamma_x+(\alpha m-(\alpha+\lambda)y)(\gamma_y-\tau_t)+(\alpha+\lambda)\gamma \tag{V}$$

$$0=\varphi_t+\frac{1}{2}x^2y\varphi_{xx}+\frac{1}{2}L^2y\varphi_{yy}+rx\varphi_x+(\alpha m-(\alpha+\lambda)y)\varphi_y-r\varphi+rV(\varphi_V-\tau_t) \tag{VI}$$

for the functions $\xi(x,y,t)$, $\gamma(x,y,t)$, $\tau(t)$, and $\varphi(x,y,t,V)=\phi(x,y,t)V+\beta(x,y,t)$. Notice that equations (IV,V and VI) are similar to the original PDE we are trying to solve. Inserting the special form of ϕ into (VI), it splits up into the following two equations

$$0=\phi_t+\frac{1}{2}x^2y\phi_{xx}+\frac{1}{2}L^2y\phi_{yy}+rx\phi_x+(\alpha m-(\alpha+\lambda)y)\phi_y-r\tau_t \tag{VI'}$$

$$0=\beta_t+\frac{1}{2}x^2y\beta_{xx}+\frac{1}{2}L^2y\beta_{yy}+rx\beta_x+(\alpha m-(\alpha+\lambda)y)\beta_y-r\beta, \tag{VI''}$$

due to the fact that φ, β, and τ_t are independent of V. The function β is independent of the other functions and equations. Furthermore, it must be a solution of the PDE (VI'') whose Lie symmetries we are looking for. So the transformation $V\mapsto V+\varepsilon\beta$ with its infinitesimal generator $X=\beta\frac{\partial}{\partial V}$ is a Lie symmetry, mapping solutions onto solution. As the Lie symmetry corresponding to the coefficient function β is as difficult to find as solving the PDE directly, it is not significant for our purpose and we do not take (VI'') into account any more.

Instead, let us focus on (II). Differentiating it two times with respect to y yields $3\gamma_{yy}+2\gamma_{yyy}=0$, whose solution is given by $\gamma=A(x,t)+B(x,t)\sqrt{y}+C(x,t)y$. Inserting this expression in (II) gives $\gamma=\tau_t y$. Hence, γ is independent of x and since (I) holds, the function ξ is independent of y, i.e. $\xi_y=\gamma_x=0$.

The same idea works with (III), whose second derivative with respect to x is $\xi_{xx}+x\xi_{xxx}=0$. The general solution to this ordinary differential equation (ODE) is $\xi=A(t)+B(t)x+C(t)x(\log(x)-1)$. Notice that (III) can only be satisfied by $\xi=A(t)+\tau_t x(\log(x)-1)$. With this knowledge, Equations (III) and (IV) simplify to

$$0=A_t+\frac{1}{2}y\tau_t+\tau_{tt}(\log(x)-1)-xy\phi_x, \tag{III'}$$

$$0=(\alpha+\lambda)\tau_t+\tau_{tt}-L^2\phi_y. \tag{IV'}$$

From (IV') one can directly derive $\phi=\frac{(\alpha+\lambda)\tau_t+\tau_{tt}}{L^2}y+D(x,t)$, as φ_y is not dependent on y. Differentiating (III') with respect to x gives

$$\tau_{tt}/x-y(D_x+xD_{xx})=0.$$

Notice that the functions τ and D are independent of y. Hence $\tau_{tt}=0$ and $D_x+xD_{xx}=0$ hold, i.e. $D(x,t)=B(t)+C(t)\log(x)$. (III') becomes $A_t+y\left(\frac{1}{2}\tau_t-C(t)\right)=0$, which means that $A_t=0$ and $C(t)=\frac{1}{2}\tau_t$. Using previous results in (VI'), we obtain

$$B_t-\tau_t\frac{r+2\alpha m(\alpha+\lambda)}{L^2}-y\tau_t\frac{1+(\alpha+\lambda)^2}{4L^2}=0.$$

As the coefficient of y must equal zero, $\tau_t=0$ holds and consequently B_t is equal to zero. Hence, the most general solution of the determining equations is

$$\xi(x,y,t,V)=c_1x$$

$$\gamma(x,y,t,V)=0$$

$$\tau(x,y,t,V)=c_2$$

$$\varphi(x,y,t,V)=c_3V+\beta(x,y,t,V)$$

where c_1, c_2 and c_3 are real constants and β is a solution of the Black-Scholes equation.

Hence, the only Lie symmetries that (2) admits have the following infinitesimal generators

$$X=c_1x\frac{\partial}{\partial x}+c_2\frac{\partial}{\partial t}+c_3V\frac{\partial}{\partial V}+\beta(x,y,t)\frac{\partial}{\partial V}.$$

Apart from the last symmetry this is a three dimensional, solvable Lie algebra, i.e. the commutator $[X,Y]=X(Y)-Y(X)$ of two arbitrary symmetries X and Y equals zero.

Next, we determine the canonical variables for the Lie symmetry with fixed constants c_1, c_2 and c_3. Therefore, we search for three functionally independent invariants $v(x,y,t,V)$, $w(x,y,t,V)$, and $u(x,y,t,V)$ that satisfy the following equation

$$X(f)=c_1x\frac{\partial f}{\partial x}+c_2\frac{\partial f}{\partial t}+c_3V\frac{\partial f}{\partial V}=0.$$

The evaluation of (7) with the choice of the following new variables $v=x\exp\left(-\frac{c_1}{c_2}t\right)$, $w=y$, $u=V\exp\left(-\frac{c_3}{c_2}t\right)$, and $\frac{\log(\ )}{}$ shows that the Lie symmetry with respect to the new variables has the desired normal symbol $\frac{\partial}{\partial}$ and that v, w, u, and s are the canonical variables.

In order to rewrite (2) in the new variables, we differentiate $V=\exp\left(\frac{c_3}{c_2}t\right)u$ with respect to x, y and t and obtain

$$V_x=\exp\left(\frac{c_3-c_1}{c_2}t\right)u_v,\qquad V_{xx}=\exp\left(\frac{c_3-2c_1}{c_2}t\right)u_{vv},$$

$$V_y=\exp\left(\frac{c_3}{c_2}t\right)u_w,\qquad V_{yy}=\exp\left(\frac{c_3}{c_2}t\right)u_{ww},$$

$$V_t=\exp\left(\frac{c_3}{c_2}t\right)\left(\frac{c_3}{c_2}u-\frac{c_1}{c_2}vu_v\right).$$

Hence, the 2-dimensional Black-Scholes equation in the new variables is given by

$$\frac{1}{2}v^2wu_{vv}+\frac{1}{2}L^2wu_{ww}+\left(r-\frac{c_1}{c_2}\right)vu_v+(\alpha m-(\alpha+\lambda)w)u_w+\left(\frac{c_3}{c_2}-r\right)u=0$$

and by setting $c_1=c_3=rc_2$ we cancel out terms with u and u_v. Therefore, the reduced Black-Scholes equation is as follows

$$\frac{1}{2}v^2wu_{vv}+\frac{1}{2}L^2wu_{ww}+(\alpha m-(\alpha+\lambda)w)u_w=0. \tag{10}$$

In order to find solutions to (10), we assume $u(v,w)=\Phi(v)\Psi(w)$ and obtain

$$\frac{1}{2}v^2w\Phi_{vv}\Psi+\frac{1}{2}L^2w\Phi\Psi_{ww}+(\alpha m-(\alpha+\lambda)w)\Phi\Psi_w=0,$$

which is equivalent to

$$\frac{L^2w\Psi_{ww}+2(\alpha m-(\alpha+\lambda)w)\Psi_w}{w\Psi}=-v^2\frac{\Phi_{vv}}{\Phi}.$$

Since the left-hand side of the equation depends only on w and the right hand side only on v, both sides must be equal to a constant C. Hence, we obtain two decoupled ordinary differential equations

$$-v^2\Phi''=C\Phi, \tag{11}$$

$$L^2w\Psi''+2(\alpha m-(\alpha+\lambda)w)\Psi'=Cw\Psi. \tag{12}$$

The general solution of equation (11) is given by

$$\Phi(v)=\begin{cases}a_1v^{\frac{1+\sqrt{1-4C}}{2}}+a_2v^{\frac{1-\sqrt{1-4C}}{2}}, & \text{if } C<\frac{1}{4},\\ a_1\sqrt{v}+a_2\sqrt{v}\log(v), & \text{if } C=\frac{1}{4},\\ a_1\sqrt{v}\cos\left(\frac{\sqrt{4C-1}}{2}\log(v)\right)+a_2\sqrt{v}\sin\left(\frac{\sqrt{4C-1}}{2}\log(v)\right), & \text{if } C>\frac{1}{4}.\end{cases}$$

Regarding the second ordinary differential equation (12), we transform it into Kummer's equation

$$wf''(w)+(b-w)f'(w)-af(w)=0,$$

by defining $\Psi(w)=w^{k_1}e^{k_2w}f(k_3w)$, where $k_1=1-\frac{2\alpha m}{L^2}$, $k_2=\frac{1}{L^2}\left(\alpha+\lambda-\sqrt{(\alpha+\lambda)^2+L^2C}\right)$, $k_3=\frac{2}{L^2}\sqrt{(\alpha+\lambda)^2+L^2C}$, $a=1-\frac{\alpha m}{L^2}+\frac{\alpha m(\alpha+\lambda)}{L^2\sqrt{(\alpha+\lambda)^2+L^2C}}$, and $b=2-\frac{2\alpha m}{L^2}$. The general solution of Kummer's equation is given by

$$f(w)=a_3M(a,b,w)+a_4U(-a,b-1,w),$$

where M and U are Kummer's functions of the first and second

Citation: Merger J, Borzi A (2016) A Lie Algebraic and Numerical Investigation of the Black-Scholes Equation with Heston Volatility Model. J Generalized Lie Theory Appl S2: 006. doi:10.4172/1736-4337.S2-006

kind, respectively. For further details, [8]. In case of $C = 0$, (12) is given by

$$L^2 w\Psi_{ww} + 2(\alpha m - (\alpha+\lambda)w)\Psi_w = 0$$

and we directly see that the first derivative of Ψ is a multiple of $w^{-\frac{2\alpha m}{L^2}} e^{\frac{2(\alpha+\lambda)}{L^2}w}$ and hence a general solution in the interval $(0, \infty)$ is given by

$$\Psi(w) = a_3 + a_4 \int_1^w s^{-\frac{2\alpha m}{L^2}} e^{\frac{2(\alpha+\lambda)}{L^2}s}\, ds.$$

Having found solutions u for (10), we obtain solutions V of (2) by applying the reverse variable transformation as follows

$$V(x,y,t) = e^{rt}u(e^{-rt}x, y). \tag{13}$$

To summarize, we obtain the following five parameter family of solutions to the two-dimensional Black-Scholes equation

$$V(x,y,t) = \begin{cases} e^{rt}\left(a_1\left(e^{-rt}x\right)^{\mu_1} + a_2\left(e^{-rt}x\right)^{\mu_2}\right)y^{k_1}e^{k_2 y}\left(a_3 M(a,b,k_3 y) + a_4 U(-a,b-1,k_3 y)\right), & \text{if } C < \frac{1}{4}, \\ \left(a_1\sqrt{e^{rt}x} + a_2\sqrt{e^{rt}x}\log(e^{-rt}x)\right)y^{k_1}e^{k_2 y}\left(a_3 M(a,b,k_3 y) + a_4 U(-a,b-1,k_3 y)\right), & \text{if } C = \frac{1}{4}, \\ \left(a_1\sqrt{e^{rt}x}\cos\left(\frac{\sqrt{4C-1}}{2}\log\left(e^{-rt}x\right)\right) + a_2\sqrt{e^{rt}x}\sin\left(\frac{\sqrt{4C-1}}{2}\log\left(e^{-rt}x\right)\right)\right) & \text{if } C > \frac{1}{4}, \\ \times y^{k_1}e^{k_2 y}\left(a_3 M(a,b,k_3 y) + a_4 U(-a,b-1,k_3 y)\right), & \\ \left(a_1 x + a_2 e^{rt}\right)\left(a_3 + a_4\int_1^y s^{-\frac{2\alpha m}{L^2}} e^{\frac{2(\alpha+\lambda)}{L^2}s}\, ds\right), & \text{if } C = 0. \end{cases} \tag{14}$$

where $\mu_{1,2} = \frac{1 \pm \sqrt{1-4C}}{2}$.

These functions do not satisfy the boundary conditions (3) given in Section 1. In order to check this we write down the boundary conditions in the new variables v, w, and u. They are given by

$$u(v,w) = e^{-rT}[a(ve^{-rT} - K)]^+, \quad \text{(i)}$$

$$u(0,w) = \frac{1-a}{2}Ke^{-rT}, \quad \text{(ii)}$$

$$\lim_{v\to\infty} \partial_v u(v,w) = \frac{1+a}{2}, \quad \text{(iii)}$$

$$\lim_{w\to\infty} u(v,w) = \begin{cases} v & \text{if } a = +1 \\ Ke^{-rT} & \text{if } a = -1. \end{cases} \quad \text{(iv)}$$

So if V would be a solution to the two-dimensional Black-Scholes equation subject to the given boundary conditions and if it corresponds to a solution u of the reduced equation, then u must satisfy the boundary conditions above. Note that, while (i) imply (ii) and (iii), (i) is not consistent with (iv). As (i) determines u and therefore u is independent of w, (iv) cannot be satisfied as $\lim_{w\to\infty} u(v,w) = e^{-rT}[a(ve^{-rT} - K)]^+$, which equals neither v nor Ke^{-rT}.

Numerical Solution of the Two-dimensional Black-Scholes Equation

This section deals with a numerical scheme to calculate an approximation to the solution of the Black-Scholes equation (2). We work with the proposal of Chang-Cooper scheme [9] and analyzed in studies of Mohammadi and Borz [10]. This disretization scheme is often used for Fokker-Planck equations, as its solutions are probability density functions and therefore are non-negative and their integral over its domain equals 1. These two properties are preserved by the Chang-Cooper (CC) difference scheme. In the case of the Black-Scholes equation the solution is also non-negative, as it models the price of an option, which must be non-negative. Hence, the choice of the CC scheme guarantees that the numerical solution will be non-negative. In order to apply the Chang-Cooper discretization scheme the two dimensional Black-Scholes equation (2) must be written in flux form. This is not possible, as the coefficient of V is $-r$ and not $-\alpha - \lambda + r - y$. However, introducing the following new variables $\tilde{x} = \log x$, $\tilde{y} = y$, $\tilde{t} = T - t$, and $\tilde{V}(\tilde{x},\tilde{y},\tilde{t}) = e^{(\alpha+\lambda-r)t}V(x,y,t)$ and computing the derivatives of V with respect to the new variables as follows

$$\begin{aligned} -\tilde{V}_{\tilde{t}} &= (\alpha+\lambda-r)e^{(\alpha+\lambda-r)t}V + e^{(\alpha+\lambda-r)t}V_t, & \tilde{V}_{\tilde{x}}\frac{1}{x} &= e^{(\alpha+\lambda-r)t}V_x, \\ \tilde{V}_{\tilde{x}\tilde{x}}\frac{1}{x^2} + \tilde{V}_{\tilde{x}}\frac{-1}{x^2} &= e^{(\alpha+\lambda-r)t}V_{xx}, & \tilde{V}_{\tilde{y}} &= e^{(\alpha+\lambda-r)t}V_y, \\ \tilde{V}_{\tilde{y}\tilde{y}} &= e^{(\alpha+\lambda-r)t}V_{yy}, \end{aligned}$$

we obtain the following PDE

$$V_t - \frac{1}{2}yV_{xx} - \frac{1}{2}L^2 yV_{yy} + \left(\frac{1}{2}y - r\right)V_x + \left((\alpha+\lambda)y - \alpha m\right)V_y + (\alpha+\lambda)V = 0. \tag{15}$$

We can write (15) in flux form as follows

$$\begin{aligned} \frac{\partial V}{\partial t} = &\frac{\partial}{\partial x}\left(B^x(x,y,t)V(x,y,t) + C^{xx}(x,y,t)\frac{\partial}{\partial x}V(x,y,t)\right) \\ &+ \frac{\partial}{\partial y}\left(B^y(x,y,t)V(x,y,t) + C^{yy}(x,y,t)\frac{\partial}{\partial y}V(x,y,t)\right) \end{aligned} \tag{16}$$

where

$$B^x(x,y,t) = r - \frac{1}{2}y, \qquad C^{xx}(x,y,t) = \frac{1}{2}y,$$

$$B^y(x,y,t) = -(\alpha+\lambda)y + \alpha m - \frac{L^2}{2}, \qquad C^{yy}(x,y,t) = \frac{L^2}{2}y.$$

At this point we would like to stress three important properties of the flux functions. To begin with, they are all independent of the time variable t. Hence, the left-hand side of the resulting linear system of equations is the same for each time iteration and the corresponding matrix must be computed only once. Moreover, both B^x and B^y are linear functions and therefore Lipschitz continuous with the constants $\gamma_x = \frac{1}{2}$ and $\gamma_y = r$. Finally, these functions must be positive in our domain. This is the case when the condition

$$0 < y < \min\left\{2r, \frac{2\alpha m - L^2}{2(\alpha+\lambda)}\right\} \tag{17}$$

is satisfied.

The transformed Black-Scholes equation (16) must be solved subject to the following transformed initial and boundary conditions:

$$\begin{aligned} V(x,y,0) &= [a(e^x - K)]^+ e^{(\alpha+\lambda-r)T}, \\ \lim_{x\to-\infty} V(x,y,t) &= \frac{1-a}{2}Ke^{(\alpha+\lambda)(T-t)-rT}, \\ \lim_{x\to\infty} \partial_x V(x,y,t)e^{-x} &= \frac{1+a}{2}e^{(\alpha+\lambda-r)(T-t)}, \\ \lim_{y\to\infty} V(x,y,t) &= \begin{cases} e^{x+(\alpha+\lambda-r)(T-t)} & \text{if } a = +1 \\ Ke^{(\alpha+\lambda)(T-t)-rT} & \text{if } a = -1. \end{cases} \end{aligned} \tag{18}$$

There are several problems that arise during implementation:

- The domain of the problem (16) subject to (18) is $\mathbb{R} \times [0,\infty) \times [0,T]$ and therefore unbounded in the space dimensions. Moreover the boundary conditions are given as a limit. For numerical purpose the domain was limited to $[x_{min}, x_{max}] \times [0, y_{max}] \times [0,T]$ and it was assumed that the function attain the limit values already at the finite boundaries.
- When $I \pm x, y$ corresponds to a point outside of the domain the values $V^{m+1}_{I\pm x,y}$ with its coefficients are added to the right hand side of the linear system of equations, as they are known.

- The boundary condition for $x \to \infty$ is given only in terms of the derivative of V with respect to x. Therefore the value V_{I+x}^{m+1} is approximated by

$$V_{I+x}^{m+1} = V_I^{m+1} + h_x \partial_x V_{I+x}^{m+1}.$$

- The derivative term is known and can be put to the right-hand side of the equation.

The values of the function on the boundary y = 0 are not given. Fortunately, as y goes to zero, B^y goes to $\alpha m - L^2/2$ and C^{yy} tends to zero. Assuming $\alpha m > L^2/2$, δ^y tends to zero, as $w = h_y B^y / C^{yy}$ goes to infinity. Hence, the coefficient of V_{I-y}^{m+1} is zero and this function values need not to be known for the calculations [9].

In the following, the numerical scheme is applied to the function type C = 0 in (14). After the variable transformation, that is used to write (2) in flux form, the test function becomes

$$f(x,y,t) = e^{(\alpha+\lambda-r)(T-t)}\left(c_1 e^x + c_2 e^{r(T-t)}\right)\left(c_3 + c_4 \int_1^y s^{\frac{-2\alpha m}{L^2}} e^{\frac{2(\alpha+\lambda)}{L^2}s} ds\right).$$

Note that this function has a singularity in y = 0, if and only if $\frac{-2\alpha m}{L^2} \le -1$. As this infinite value might arise problems while numerical calculations, the set of parameters is chosen such that $2\alpha m < L^2$ holds. In particular, the test function was calculated in the domain $[-0.5, 0.5] \times [0,1]$ with the following parameters

$$r = 0.02, \quad m = 0.2, \quad \alpha = 1, \quad \lambda = 1, \quad L = 1, \quad a = 1, \quad K = 1, \quad T = 1.$$

Unfortunately, it is not possible to chose a set of parameters such that the test function has no singularity at y = 0 and additionally there exists a domain where all flux functions are positive. That is because $2\alpha m < L^2$ imply $\frac{2\alpha m - L^2}{2(\alpha+\lambda)} < 0$ and therefore the necessary condition for (17) is not fulfilled. Consequently, there is no proof in this case, that the numerical solution is positive. Nevertheless, the convergence order can be observed. Figure 1 shows the difference of the numerical solution to the exact test function in terms of the norm

$$\|f\| = \sqrt{h_x h_y \delta t \sum_{I,m} \left|f_I^m\right|^2}.$$

The plot data is shown in Table 1 where N, M and Q is the number of grid points in the x-, y- and t-dimension, respectively. A small time-step size is used in order to have a small error for the time discretization and to investigate the dependence of the error on the spatial-grid size.

The numerical experiments show that the discretization that is used provides only first-order convergence, i.e., doubling the grid point number in each spatial dimension and therefore halving the grid size h results in an error that is half as big as before. Notice that second-order convergence is proven in literature of Mohammadi and Borz [10] with the assumption of zero boundary conditions. In order to validate this theoretical result the same procedure is done with a Gaussian bell function that is almost zero on the boundaries:

$$\Phi(x,y,t) = \exp\left(-t - (6x)^2 - (6y-3)^2\right)$$

In contrast to the test function f, this function Φ does not satisfy the partial differential equation (16). Hence the deviation to the PDE

$$H(\Phi) := \Phi_t - \frac{1}{2} y \Phi_{xx} - \frac{1}{2} L^2 y \Phi_{yy} + \left(\frac{1}{2} y - r\right)\Phi_x + \left((\alpha+\lambda) y - \alpha m\right)\Phi_y + (\alpha+\lambda)\Phi$$

is added to the right hand side of the linear system of equations according to

$$3\Phi_I^m - \frac{2\delta t}{h}\sum_{i=1}^n \left(F_{I+1_i/2}^i - F_{I-1_i/2}^i\right) = 4\Phi_I^{m-1} - \Phi_I^{m-2} + H(\Phi)_I^m.$$

Numerical experiments are performed with the grid sizes according to Table 2.

Φ was approximated in the domain $[-0.5, 0.5] \times [0,1]$ with the parameters being

$$r = 1, \quad m = 3, \quad \alpha = 2.5, \quad \lambda = 1, \quad L = 1, \quad a = 1, \quad K = 1, \quad T = 1.$$

Here, the flux functions are positive, as $2r = \frac{2\alpha m - L^2}{2(\alpha+\lambda)} = 2$ With the logarithmic plot of the error in Figure 2 we can see that second-order convergence is obtained.

In the following, we use the Chang-Cooper numerical scheme to calculate a numerical solution. Here, this numerical solution in the case of a call option is compared to the solution of the Black-Scholes equation, where the volatility is assumed to be constant. It is given by

$$V(S,t) = a\left(S\Phi(ad_1) - Ke^{-r(T-t)}\Phi(ad_2)\right) \tag{19}$$

$$d_{1,2} = \frac{\ln(S/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

where Φ is the cumulative distribution function of a normally distributed random variable with mean 0 and variance 1. This function is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-s^2/2} ds.$$

The spatial domain of discretization is $[-4, 0.5] \times [0,2]$. After reversing the variable transformation the option price can be evaluated for $x \in [0.0183, 1.6487]$ and $y \in [0,2]$. The following parameters are used

$$r = 0.06, \quad m = 0.5, \quad \alpha = 2, \quad \lambda = 1, \quad L = 1, \quad a = 1, \quad K = 1, \quad T = 1.$$

The model constant m represents the square of the average volatility and the stochastic process tends to this value. Hence, if one starts the process with the value y = 0.5, the stochastic process for the volatility $\sqrt{v_t}$ is likely to be almost constant to $\sqrt{0.5}$. As you can see in the lower left plot of Figure 3, the calculated price is nearly the same with both models. In contrast, regarding the case of a currently volatility lower than $\sqrt{0.5}$ the price of the option calculated with the extended Black-Scholes equation is higher than that of the model with constant volatility, because it takes into account that the volatility will rise. This

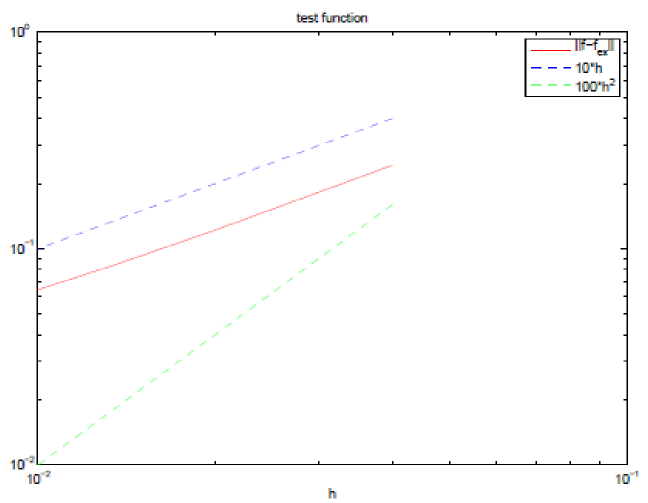

Figure 1: Logarithmic plot of the norm of the error || *f* -*fex* ||.

N	26	51	76	101
M	26	51	76	101
Q	500	500	500	500
$h_x = h_y$	1/25	1/50	1/75	1/100
$\|\| f - f_{ex} \|\|$	0.243	0.122	0.083	0.065

Table 1: Numerical error for different grid sizes.

N	26	51	76	101
M	26	51	76	101
Q	500	500	500	500
$h_x = h_y$	1/25	1/50	1/75	1/100
$\|\| \Phi - \Phi_{ex} \|\|$	$3.25*10^{3}$	$0.83*10^{-3}$	$0.37*10^{-3}$	$0.21*10^{-3}$

Table 2: Numerical error for different grid sizes.

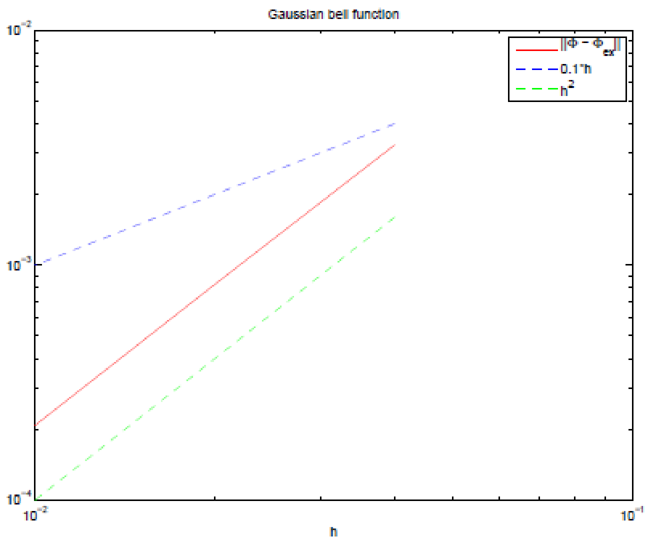

Figure 2: Logarithmic plot of the norm of the error $|| \Phi-\Phi_{ex}||$ against the spatial mesh size *h*.

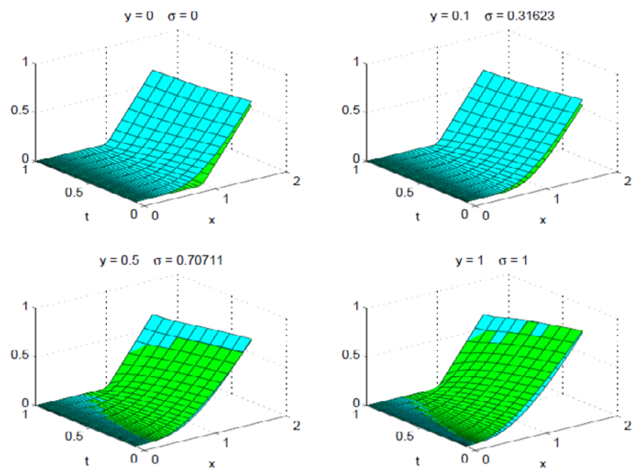

Figure 3: Comparison between the solutions of the Black-Scholes equation with fixed (green) and variable (cyan) volatility.

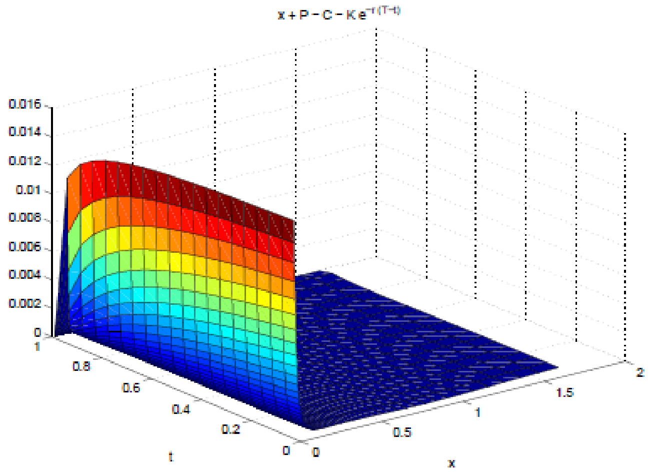

Figure 4: Arithmetic mean over the y-dimension of the absolute deviation from the Put-Call-Parity equation.

effect can be seen in upper left and right plots. Finally, in the lower right plot the simpler model overestimates the price, when the initial volatility is higher than $\sqrt{0.5}$, due to the fact that it is likely to fall.

In addition, the numerical solution satisfies the so-called Put-Call-Parity. The price of a call option *C* and the price of a put option *P* subject to the same asset with price *x*, that have the strike price *K* and the expiry date *T* in common, are related by the following formula [3]

$$x + P - C = Ke^{-r(T-t)}.$$

We compute also the price for the put option and observe the absolute deviation for the Put-Call-Parity formula that we average along the y-dimension. Figure 4 shows the result depending on *x* and *t*. Apart from small *x* values the error is in the range of the numerical error of the Chang-Cooper-Scheme. The drastic increase of the error for $x \to 0$ is due to the fact, that the boundary condition for $\tilde{x} = -\infty$ for the Black-Scholes equation in flux form is applied at the finite value $\tilde{x} = -4$. This corresponds to $x = e^{-4}$ as the transformation was $\tilde{x} = \log(x)$. Consequently, the numerical solution for the put option takes the value $Ke^{-r(T-t)}$ at $x = e^{-4}$, whereas the correct value is $Ke^{-r(T-t)} - e^{-4}$, as *C* tends to zero as *x* goes to zero and therefore the price for the put option is $P = Ke^{-r(T-t)} - x$. To conclude, it is evident why there is such a great error for small *x*, and moreover it is not relevant as *x* gets never so small in applications.

Conclusion

The aim of this work was to solve the partial differential Black-Scholes equation with Heston volatility model. Therefore, an analytical technique due to Sophus Lie that can be use to reduce the number of independent variables of a partial differential equation was presented and applied to the Black-Scholes equation. A five-parameter family of solutions was found. These functions do not satisfy the boundary conditions of the option price problem and henceforth numerical schemes are necessary to obtain approximate solutions. In the last part of this work the Chang-Cooper discretization scheme was used to calculate the option price function numerically. Its convergence was tested with an exact solution of the PDE, which was found by the Lie theoretical analysis. Finally, the numerical scheme was applied to compute the price of an option and good result were obtained in accordance with economic reasoning.

Acknowledgement

This work was supported in part by project "Multi - ITN STRIKE - Novel Methods in Computational Finance" by EU Grant Agreement Nr. 304617 and by BMBF project 05M2013 'ROENOBIO: Robust energy optimization of fermentation processes for the production of biogas and wine.

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This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, University of Antananarivo, Madagascar

Citation: Merger J, Borzi A (2016) A Lie Algebraic and Numerical Investigation of the Black-Scholes Equation with Heston Volatility Model. J Generalized Lie Theory Appl S2: 006. doi:10.4172/1736-4337.S2-006

Journal of Generalized Lie Theory and Applications
ISSN: 1736-4337
Nikolaev, J Generalized Lie Theory Appl 2015, S2
http://dx.doi.org/10.4172/1736-4337.S2-007
Research Article Open Access

Real Multiplication Revisited

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Abstract

It is proved that the Hilbert class field of a real quadratic field $Q(\sqrt{D})$ modulo a power *m* of the conductor *f* is generated by the Fourier coefficients of the Hecke eigenform for a congruence subgroup of level *fD*.

Keywords: Class field; Real multiplication

Introduction

The Kronecker's *Jugendtraum* is a conjecture that the maximal unramified abelian extension (The Hilbert class field) of any algebraic number field is generated by the special values of modular functions attached to an abelian variety. The conjecture is true for the rational field and imaginary quadratic fields with the modular functions being an exponent and the j-invariant, respectively. In the case of an arbitrary number field, a description of the abelian extensions is given by class field theory, but an explicit formula for the generators of these abelian extensions, in the sense sought by Kronecker, is unknown even for the real quadratic fields.

The problem was first studied by Hecke [1]. A description of abelian extensions of real quadratic number fields in terms of coordinates of points of finite order on abelian varieties associated with certain modular curves was obtained in studies of Shimura [2]. Stark formulated a number of conjectures on abelian extension of arbitrary number fields, which in the real quadratic case amount to specifying generators of these extensions using special values of Artin L-functions [3]. Based on an analogy with complex multiplication, Manin suggested to use the so-called "pseudo-lattices" $\mathbb{Z}+\mathbb{Z}\theta$ in $\mathbb{R}$ having non-trivial real multiplications to produce abelian extensions of real quadratic fields [4]. Similar to the case of complex multiplication, the endomorphism ring $\mathfrak{R}_{\mathfrak{f}}=\mathbb{Z}+\mathfrak{f}O_{\mathfrak{k}}$ of pseudo-lattice $\mathbb{Z}+\mathbb{Z}\theta$ is an order in the real quadratic field $\mathfrak{k}=\mathbb{Q}(\theta)$, where $O_{\mathfrak{k}}$ is the ring of integers of $\mathfrak{k}$ and $\mathfrak{f}$ is the conductor of $\mathfrak{R}_{\mathfrak{f}}$; Manin calls these pseudo-lattices with *real multiplication*.

The aim of our note is a formula for generators of the Hilbert class field of real quadratic fields based on a modularity and a symmetry of complex and real multiplication. To give an idea, let

$$\Gamma_1(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\mid a\equiv d\equiv 1 \mod N, c\equiv 0 \mod N\right\} \tag{1}$$

be a congruence subgroup of level $N\geq 1$ and $\mathbb{H}$ be the Lobachevsky half-plane; let $X_1(N):=\mathbb{H}/\Gamma_1(N)$ be the corresponding modular curve and $S_2(\Gamma_1(N))$ the space of all cusp forms on $\Gamma_1(N)$ of weight 2. Let $\mathcal{E}_{CM}^{(-D,f)}$ be elliptic curve with complex multiplication by an order $\mathfrak{R}_{\mathfrak{f}}=\mathbb{Z}+\mathfrak{f}O_{\mathfrak{k}}$ in the field $\mathfrak{k}=\mathbb{Q}(\sqrt{-D})$ [5]. Denote by $\mathcal{K}^{ab}(k):=k(j(\mathcal{E}_{CM}^{(-D,f)}))$ the Hilbert class field of k modulo conductor $f\geq 1$ and let $N=fD$; let $Jac(X_1(fD))$ be the Jacobian of modular curve $X_1(fD)$. There exists an abelian sub-variety $A_\phi\subset Jac(X_1(fD))$, such that its points of finite order generate $\mathbb{K}^{ab}(k)$, [2,6,7], Section 8. The $\mathbb{K}^{ab}(k)$ is a *CM-field*, i.e. a totally imaginary quadratic extension of the totally real field $\mathbb{K}_\varphi$ generated by the Fourier coefficients of the Hecke eigenform $\phi(z)\in S_2(\Gamma_1(fD))$ [2]. In particular, there exists a holomorphic map $X_1^0(fD)\to\mathcal{E}_{CM}^{(-D,f)}$, where $X_1^0(fD)$ is a Riemann surface such that $Jac(X_1^0(fD))\cong A_\phi$; we refer to the above as a *modularity* of complex multiplication.

Recall that (twisted homogeneous) coordinate ring of an elliptic curve $\varepsilon(\mathbb{C})$ is isomorphic to a *Sklyanin algebra*, [8]; the norm-closure of a self-adjoint representation of the Sklyanin algebra by the linear operators on a Hilbert space $\mathcal{H}$ is isomorphic to a noncommutative torus $\mathcal{A}_\theta$, [9] for the definition.

Whenever elliptic curve $\mathcal{E}(C)\cong\mathcal{E}_{CM}^{(-D,f)}$ has complex multiplication, the noncommutative torus $\mathcal{A}_\theta$ has real multiplication by an order $\mathfrak{R}_{\mathfrak{f}}=\mathbb{Z}+\mathfrak{f}O_{\mathfrak{k}}$ in the field $\mathfrak{k}=\mathbb{Q}(\sqrt{-D})$; moreover, it is known that $\mathfrak{f}=f^m$ for the minimal power m satisfying an isomorphism:

$$Cl(\mathfrak{R}_{f^m})\cong Cl(R_f), \tag{2}$$

where $Cl(R_f)$ and $Cl(\mathfrak{R}_{\mathfrak{f}})$ are the ideal class groups of orders R_f and $\mathfrak{R}_{\mathfrak{f}}$, respectively. We shall refer to (2) as a *symmetry* of complex and real multiplication. The noncommutative torus with real multiplication by $\mathfrak{R}_{\mathfrak{f}}$ will be denoted by $\mathcal{A}_{RM}^{(D,\mathfrak{f})}$.

Remark 1: The isomorphism (2) can be calculated using the well-known formula for the class number of a non-maximal order $\mathbb{Z}+fO_K$ of a quadratic field $K=\mathbb{Q}(\sqrt{D})$:

$$h_{\mathbb{Z}+fO_K}=\frac{h_{O_K}f}{e_f}\prod_{p|f}\left(1-\left(\frac{D}{p}\right)\frac{1}{p}\right), \tag{3}$$

where h_{O_K} is the class number of the maximal order O_K, e_f is the index of the group of units of $\mathbb{Z}+fO_K$ in the group of units of O_K, p is a prime number and $\frac{D}{p}$ is the Legendre symbol [10,11].

The (twisted homogeneous) coordinate ring of the Riemann surface $X_1^0(fD)$ is an *AF-algebra* $\mathbb{A}_{\phi^0}$ linked to a holomorphic differential $\phi^0(z)dz$ on $X_1^0(fD)$, see Section 2.2, Definition 1 and Remark 5 for the details; the Grothendieck semigroup $K_0^+(\mathbb{A}_{\phi^0})$ is a pseudo-lattice $\mathbb{Z}+\mathbb{Z}\theta_1+\ldots+\mathbb{Z}\theta_{n-1}$ in the number field $\mathbb{K}_\varphi$, where n equals the genus of $X_1^0(fD)$. Moreover, a holomorphic map $X_1^0(fD)\to\mathcal{E}_{CM}^{(-D,f)}$ induces the C^*-algebra homomorphism $\mathbb{A}_{\phi^0}\to\mathcal{A}_{RM}^{(D,\mathfrak{f})}$ between the corresponding coordinate rings, so that the following diagram commutes:

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Received December 22, 2015; **Accepted** January 25, 2016; **Published** January 28, 2016

Citation: Nikolaev IV (2016) Real Multiplication Revisited. J Generalized Lie Theory Appl S2: 007. doi:10.4172/1736-4337.S2-007

Citation: Nikolaev IV (2016) Real Multiplication Revisited. J Generalized Lie Theory Appl S2: 007. doi:10.4172/1736-4337.S2-007

$$\begin{tikzcd} X_1^0(fD) \arrow[r, "\text{coordinate}", "\text{map}"'] \arrow[d] & \mathbb{A}_{\phi^0} \arrow[d] \\ \mathcal{E}_{CM}^{(-D,f)} \arrow[r, "\text{coordinate}", "\text{map}"'] & \mathcal{A}_{\mathrm{RM}}^{(D,\mathfrak{f})} \end{tikzcd}$$

But $K_0^+(\mathcal{A}_{RM}^{(D,\mathfrak{f})})$ is a pseudo-lattice $\mathbb{Z}+\mathbb{Z}\theta$ in the field $\mathfrak{k}$, such that $End\ (\mathbb{Z}+\mathbb{Z}\theta) \cong \mathfrak{R}_f$; in other words, one can use the above diagram to control the arithmetic of the field $\mathcal{K}_\varphi$ by such of the real quadratic field $\mathfrak{k}$. Roughly speaking, this observation solves the Kronecker's Jugendtraum for the real quadratic fields; namely, the following is true.

Theorem 1. *The Hilbert class field of a real quadratic field* $\mathfrak{k}=\mathbb{Q}(\sqrt{D})$ *modulo conductor* f^m *is an extension of* $\mathfrak{k}$ *by the Fourier coefficients of the Hecke eigenform* $\phi(z)\in S_2(\Gamma_1(fD))$, *where m is the smallest positive integer satisfying isomorphism (2).*

Remark 2. Theorem 1 can be used to compute concrete extensions. For instance, theorem 1 says that for the quadratic field $\mathbb{Q}\sqrt{15}$ its Hilbert class field is isomorphic to $\mathbb{Q}\sqrt{-1+\sqrt{15}}$ and for $\mathbb{Q}\sqrt{14}$ such a field modulo conductor $\mathfrak{f}$ = 8 is isomorphic to $\mathbb{Q}\left(\sqrt[4]{-27+8\sqrt{14}}\right)$ see section 4 for more examples.

The article is organized as follows. Section 2 covers basic facts on real multiplication and AF-algebras of the Hecke eigenforms. Theorem 1 is proved in Section 3. Section 4 contains numerical examples illustrating theorem 1.

Preliminaries

The reader can find basics of the C^*-algebras in studies of Murphy [12] and their K-theory in theory of Blackadar [13]. The noncommutative tori are covered in literature of Rieffel [9] and real multiplication in studies of Manin [4]. For main ideas of non-commutative algebraic geometry, see the survey by Stafford and van den Bergh [8]. The AF-algebras are reviewed in studies of Effros [14]. For a general theory of modular forms we refer to literature of Diamond and Shurman [15].

Real multiplication

The noncommutative torus $\mathcal{A}_\theta$ is a universal C^**-algebra* generated by the unitary operators u and v acting on a Hilbert space $\mathcal{H}$ and satisfying the commutation relation $vu=e^{2\pi i\theta}uv$, where θ is a real number. The C^*-algebra $\mathcal{A}_\theta$ is said to be stably isomorphic (Morita equivalent) to $\mathcal{A}_{\theta'}$, whenever $\mathcal{A}_\theta\otimes\mathcal{K}\cong\mathcal{A}_{\theta'}\otimes\mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of all compact operators on $\mathcal{H}$; the $\mathcal{A}_\theta$ is stably isomorphic to $\mathcal{A}_{\theta'}$ if and only if

$$\theta'=\frac{a\theta+b}{c\theta+d} \text{ for some matrix } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{Z}) \tag{4}$$

The K-theory of $\mathcal{A}_\theta$ is two-periodic and $K_0(\mathcal{A}_\theta)\cong K_1(\mathcal{A}_\theta)\cong\mathbb{Z}^2$ so that the Grothendieck semigroup $K_0^+(\mathcal{A}_\theta)$ corresponds to positive reals of the pseudo-lattice $\mathbb{Z}+\mathbb{Z}\theta\subset\mathbb{R}$. The $\mathcal{A}_\theta$ is said to have *real multiplication*, if θ is a quadratic irrationality, i.e. irrational root of a quadratic polynomial in $\mathbb{Z}[x]$. The real multiplication says that the endomorphism ring of pseudo-lattice $\mathbb{Z}+\mathbb{Z}\theta$ exceeds the ring $\mathbb{Z}$ corresponding to multiplication by m endomorphisms; similar to complex multiplication, it means that the endomorphism ring is isomorphic to an order $\mathfrak{R}_\mathfrak{f}=\mathbb{Z}+\mathfrak{f}O_\mathfrak{k}$ of conductor $f\ \geq 1$ in the real quadratic field k = $\mathbb{K}(\theta)$, hence the name. If $D>0$ is the discriminant of k, then by $\mathcal{A}_{RM}^{(D,\mathfrak{f})}$ we denote torus $\mathbb{A}_\theta$ with real multiplication by the order $\mathfrak{R}_\mathfrak{f}$.

The Sklyanin algebra $S_{\alpha,\beta,\gamma}(\mathbb{C})$ is a free C-algebra on four generators and six relations:

$$\begin{cases} x_1x_2-x_2x_1 &=& \alpha(x_3x_4+x_4x_3),\\ x_1x_2+x_2x_1 &=& x_3x_4-x_4x_3,\\ x_1x_3-x_3x_1 &=& \beta(x_4x_2+x_2x_4),\\ x_1x_3+x_3x_1 &=& x_4x_2-x_2x_4,\\ x_1x_4-x_4x_1 &=& \gamma(x_2x_3+x_3x_2),\\ x_1x_4+x_4x_1 &=& x_2x_3-x_3x_2, \end{cases} \tag{5}$$

where $\alpha+\beta+\gamma+\alpha\beta\gamma=0$; such an algebra corresponds to a *twisted homogeneous coordinate ring* of an elliptic curve in the complex projective space $\mathbb{C}P^3$ given by the intersection of two quadric surfaces of the form $\mathcal{E}_{\alpha,\beta,\gamma}(\mathbb{C})=\{(u,v,w,z)\in\mathbb{C}P^3\mid u^2+v^2+w^2+z^2=\frac{1-\alpha}{1+\beta}v^2+\frac{1+\alpha}{1-\gamma}w^2+z^2=0\}$.

Being such a ring means that the algebra $S_{\alpha,\beta,\gamma}$ satisfies an isomorphism

$$\mathbf{Mod}\ (\ {}_{\alpha,\beta,\gamma}(\mathbb{C}))/\ \mathbf{Tors}\quad \mathbf{Coh}\ (\ {}_{\alpha,\beta,\gamma}(\mathbb{C})), \tag{6}$$

where **Coh** is the category of quasi-coherent sheaves on $\mathcal{E}_{\alpha,\beta,\gamma}(\mathbb{C})$, **Mod** the category of graded left modules over the graded ring $S_{\alpha,\beta,\gamma}(\mathbb{C})$ and **Tors** the full sub-category of **Mod** consisting of the torsion modules, [8].

If one sets $x_1=u, x_2=u^*, x_3=v, x_4=v^*$, then there exists a self-adjoint representation of the Sklyanin $*$-algebra $S_{\alpha,1,-1}(\mathbb{C})$ by linear operators on a Hilbert space $\mathcal{H}$, such that its norm-closure is isomorphic to $\mathcal{A}_\theta$; namely, $\mathcal{A}_\theta^0\cong S_{\alpha,1,-1}(\mathbb{C})/I_\mu$ where $\mathcal{A}_\theta^0$ is a dense sub-algebra of $\mathcal{A}_\theta$ and I_μ is an ideal generated by the "scaled unit" relations $x_1x_3=x_3x_4=\frac{1}{\mu}e$, where $\mu>0$ is a constant. Thus the algebra $\mathcal{A}_\theta$ is a coordinate ring of elliptic curve $E(C)$, such that isomorphic elliptic curves correspond to the stably isomorphic (Morita equivalent) noncommutative tori; this fact explains the modular transformation law in (4). In particular, if ε (ℂ) has complex multiplication by an order $\mathfrak{R}_\mathfrak{f}=\mathbb{Z}+\mathfrak{f}O_\mathfrak{k}$ in a quadratic field $\mathfrak{k}=\mathbb{Q}(\sqrt{-D})$, then $\mathcal{A}_\theta$ has real multiplication by an order $\mathfrak{R}_\mathfrak{f}=\mathbb{Z}+\mathfrak{f}O_\mathfrak{k}$ in the quadratic field $\mathfrak{k}=\mathbb{Q}(\sqrt{D})$, where f is the smallest integer satisfying an isomorphism $Cl\,(\mathfrak{R}_\mathfrak{f})\cong Cl\,(R_f)$, [16]; the isomorphism is a necessary and sufficient condition for $\mathcal{A}_{RM}^{(D,\mathfrak{f})}$ to discern non-isomorphic elliptic curves $\mathcal{E}_{CM}^{(-D,f)}$ having the same endomorphism ring R_f. For the constraint $\mathfrak{f}=f^m$, see remark 6.

AF-algebra of the Hecke eigenform

An *AF-algebra* (Approximately Finite C^*-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional C^*-algebras M_n, where M_n is the C^*-algebra of the $n\times n$ matrices with entries in ℂ. Here the index $n=(n_1,\dots,n_k)$ represents the semi-simple matrix algebra $M_n=M_{n_1}\oplus\dots\oplus M_{n_k}$. The ascending sequence mentioned above can be written as $M_1\xrightarrow{\varphi_1}M_2\xrightarrow{\varphi_2}\dots$, where M_i are the finite dimensional C^*-algebras and ϕ_i the homomorphisms between such algebras. The homomorphisms ϕ_i can be arranged into a graph as follows. Let $M_i=M_{i_1}\oplus\dots\oplus M_{i_k}$ and $M_{i'}=M_{i_1'}\oplus\dots\oplus M_{i_{k'}}$ be the semi-simple C^*-algebras and $\varphi_i: M_i\to M_{i'}$ the homomorphism. One has two sets of vertices $V_{i_1},\dots,V_{i_k}$ and $V_{i_1'},\dots,V_{i_{k'}}$ joined by b_{rs} edges whenever the summand M_{i_r} contains b_{rs} copies of the summand $M_{i_s'}$ under the embedding ϕ_i. As i varies, one obtains an infinite graph called the *Bratteli diagram* of the AF-algebra. The matrix $B=(b_{rs})$ is known as a *partial multiplicity matrix*; an infinite sequence of B_i defines a unique AF-algebra. An AF-algebra is called *stationary* if $B_i=Const=B$, [14], when two non-similar matrices B and B' have the same characteristic polynomial, the corresponding stationary AF-algebras will be called *companion AF-algebras*.

Citation: Nikolaev IV (2016) Real Multiplication Revisited. J Generalized Lie Theory Appl S2: 007. doi:10.4172/1736-4337.S2-007

Let $N \geq 1$ be a natural number and consider a (finite index) subgroup of the modular group given by the formula:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \mod N, c \equiv 0 \mod N \right\}. \quad (7)$$

Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane and let $\Gamma_1(N)$ act on $\mathbb{H}$ by the linear fractional transformations; consider an orbifold $\mathbb{H}/\Gamma_1(N)$. To compactify the orbifold at the cusps, one adds a boundary to $\mathbb{H}$, so that $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ and the compact Riemann surface $X_1(N) = \mathbb{H}^*/\Gamma_1(N)$ is called a *modular curve*. The meromorphic functions (z) on $\mathbb{H}$ that vanish at the cusps and such that

$$\phi\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 \phi(z), \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N), \quad (8)$$

are called *cusp forms* of weight two; the (complex linear) space of such forms will be denoted by $S_2(\Gamma_1(N))$. The formula $\phi(z) \mapsto \omega = \phi(z)dz$ defines an isomorphism $S_2(\Gamma_1(N)) \cong \Omega_{hol}(X_1(N))$, where $\Omega_{hol}(X_1(N))$ is the space of all holomorphic differentials on the Riemann surface $X_1(N)$. Note that $\dim_{\mathbb{C}}(S_2(\Gamma_1(N)) = \dim_{\mathbb{C}}(\Omega_{hol}(X_1(N)) = g$, where $g = g(N)$ is the genus of the surface $X_1(N)$. A *Hecke operator*, T_n, acts on $S_2(\Gamma_1(N))$ by the formula $T_n\phi = \sum_{m \in Z} \gamma(m) q^m$, where $\gamma(m) = \sum_{a \mid \mathbf{GCD}(m,n)} ac_{mn/a^2}$ and $\phi(z) = \sum_{m \in \mathbb{Z}} c(m) q^m$ is the Fourier series of the cusp form ϕ at $q = e^{2\pi i z}$. Further, T_n is a self-adjoint linear operator on the vector space $S_2(\Gamma_1(N))$ endowed with the Petersson inner product; the algebra $\mathbb{T}_N := \mathbb{Z}[T_1, T_2, \ldots]$ is a commutative algebra. Any cusp form $\phi \in S_2(\Gamma_1(N))$ that is an eigenvector for one (and hence all) of T_n, is referred to as a *Hecke eigenform*. The Fourier coefficients $c(m)$ of ϕ are algebraic integers, and we denote by $K_\phi = Q(c(m))$ an extension of the field Q by the Fourier coefficients of ϕ. Then $\mathcal{K}_\varphi$ is a real algebraic number field of degree $1 \leq \deg(\mathcal{K}_\phi \mid \mathbb{Q}) \leq g$, where g is the genus of the surface $X_1(N)$ [5], Proposition 6.6.4. Any embedding $\sigma: \mathcal{K}_\phi \to \mathbb{C}$ conjugates ϕ by acting on its coefficients; we write the corresponding Hecke eigenform $\phi^\sigma(z) := \sum_{m \in Z} \sigma(c(m)) q^m$ and call ϕ^σ a *conjugate* of the Hecke eigenform ϕ.

Let $\omega = \phi(z)dz \in \Omega_{hol}(X)$ be a holomorphic differential on a Riemann surface X. We shall denote by R(w) a closed form on X (the real part of w) and consider its periods $\lambda_i = \int_{\gamma_i} \Re(\omega)$ against a basis γ_i in the (relative) homology group $H_1(X, Z(\Re(\omega)); \mathbb{Z})$, where $Z(\Re(\omega))$ is the set of zeros of the form R(w). Assume $\lambda_i > 0$ and consider the vector $\theta = (\theta_1, \ldots, \theta_{n-1})$ with $\theta_i = \lambda_{i+1}/\lambda_1$. The *Jacobi-Perron continued fraction* of θ is given by the formula:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{i \to \infty} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_i \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix} = \lim_{i \to \infty} B_i \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, \quad (9)$$

where $b_i = (b_1^{(i)}, \ldots, b_{n-1}^{(i)})^T$ is a vector of non-negative integers, I is the unit matrix and $\mathbb{I} = (0, \ldots, 0, 1)^T$ [17]. By $\mathbb{A}_\phi$ we shall understand the AF-algebra given the Bratteli diagram with partial multiplicity matrices B_i. If $\phi(z) \in S_2(\Gamma_1(N))$ is a Hecke eigenform, then the corresponding AF-algebra $\mathbb{A}_\phi$ is *stationary* with the partial multiplicity matrices $B_i = Const = B$; moreover, each conjugate eigenform ϕ^σ defines a *companion* AF-algebra A_{ϕ^σ}. It is known that $K_0^+(\mathbb{A}_\phi) \cong \mathbb{Z} + \mathbb{Z}\theta_1 + \ldots + \mathbb{Z}\theta_{n-1} \subset \mathcal{K}_\phi$, where $\mathcal{K}_\varphi$ is an algebraic number field generated by the Fourier coefficients of ϕ, [18].

Proof of Theorem 1

Definition 1. *Let* $A_\phi \subset Jac(X_1(fD))$ *be an abelian variety associated to the Hecke eigenform* $\phi(z) \in S_2(\Gamma_1(fD))$ [15], *Definition 6.6.3. By* $X_1^0(fD)$ *we shall understand the Riemann surface of genus g, such that*

$$Jac(X_1^0(fD)) \cong A_\phi. \quad (10)$$

By $\phi^0(z)dz \in \Omega_{hol}(X_1^0(fD))$ *we denote the image of the Hecke eigenform* $\phi(z)dz \in \Omega_{hol}(X_1(fD))$ *under the holomorphic map* $X_1(fD) \to X_1^0(fD)$.

Remark 3. The surface $X_1^0(fD)$ is correctly dened. Indeed, since the abelian variety A_φ is the product of g copies of an elliptic curve with the complex multiplication, there exists a holomorphic map from A_φ to the elliptic curve. For a Riemann surface X of genus g covering the elliptic curve ε_{CM} by a holomorphic map (such a surface and a map always exist), one gets a period map $X \to A_\varphi$ by closing the arrows of a commutative diagram $A_\varphi \to \varepsilon_{CM} \leftarrow X$. It is easy to see, that the Jacobian of X coincides with A_φ and we set $X_1^0(fD) := X$.

Lemma 1. $g(X_1^0(fD)) = \deg(\mathcal{K}^{ab}(k) \mid k)$.

Proof. By definition, abelian variety A_φ is the quotient of $\mathbb{C}^n$ by a lattice of periods of the Hecke eigenform $\phi(z) \in S_2(\Gamma_1(fD))$ and all its conjugates $\phi^\sigma(z)$ on the Riemann surface $X_1(fD)$. These periods are complex algebraic numbers generating the Hilbert class field $K^{ab}(k)$ over imaginary quadratic field $k = Q(\sqrt{-D})$ modulo conductor f, [2,6,7], Section 8. The number of linearly independent periods is equal to the total number of the conjugate eigenforms $\phi^\sigma(z)$, i.e. $|\sigma| = n = \dim_{\mathbb{C}}(A_\phi)$. Since real dimension $\dim_{\mathbb{R}}(A_\phi) = 2n$, we conclude that $\deg(\mathcal{K}^{ab}(k) \mid \mathbb{Q}) = 2n$ and, therefore, $\deg(\mathcal{K}^{ab}(k) \mid k) = n$. But $\dim_{\mathbb{C}}(A_\phi) = g(X_1^0(fD))$ and one gets $g(X_1^0(fD)) = \deg(\mathcal{K}^{ab}(k) \mid k)$. Lemma 1 follows.

Corollary 1. $g(X_1^0(fD)) = |Cl(R_f)|$.

Proof. Because $\mathcal{K}^{ab}(k)$ is the Hilbert class field over k modulo conductor f, we must have

$$Gal(\mathcal{K}^{ab}(k) \mid k) \cong Cl(R_f), \quad (11)$$

where Gal is the Galois group of the extension $\mathcal{K}^{ab}(k) \mid k$ and $Cl(R_f)$ is the class group of ring R_f, [5]. But $|Gal(\mathcal{K}^{ab}(k) \mid k)| = \deg(\mathcal{K}^{ab}(k) \mid k)$ and by lemma 1 we have $\deg(\mathcal{K}^{ab}(k) \mid k) = g(X_1^0(fD))$. In view of this and isomorphism (11), one gets $|Cl(R_f)| = |Gal(\mathcal{K}^{ab} \mid k)| = g(X_1^0(fD))$. Corollary 1 follows.

Lemma 2. $g(X_1^0(fD)) = \deg(\mathcal{K}_\phi \mid Q)$.

Proof. It is known that $\dim_{\mathbb{C}}(A_\phi) = \deg(\mathcal{K}_\phi \mid \mathbb{Q})$ [15], Proposition 6.6.4. But abelian variety $A_\phi \cong Jac(X_1^0(fD))$ and, therefore, $\dim_{\mathbb{C}}(A_\phi) = \dim_{\mathbb{C}}(Jac(X_1^0(fD))) = g(X_1^0(fD))$, hence the lemma.

Corollary 2. $\deg(\mathcal{K}_\phi \mid \mathbb{Q}) = |Cl(\mathfrak{R}_\mathfrak{f})|$.

Proof. From lemma 2 and corollary 1 one gets $\deg(\mathcal{K}_\phi \mid \mathbb{Q}) = |Cl(R_f)|$. In view of this and equality (2), one gets the conclusion of corollary 2.

Lemma 3. (Basic lemma) $Gal(\mathcal{K}_\phi \mid \mathbb{Q}) \cong Cl(\mathfrak{R}_\mathfrak{f})$.

Proof. Let us outline the proof. In view of lemma 2 and corollaries 1-2, we denote by h the single integer $g(X_1^0(fD)) = |Cl(R_f)| = |Cl(\mathfrak{R}_1)| = \deg(\mathcal{K}_\phi \mid \mathbb{Q})$. Since $\deg(\mathcal{K}_\phi \mid \mathbb{Q}) = h$, there exist $\{\phi_1, \ldots, \phi_h\}$ conjugate Hecke eigenforms $\phi_i(z) \in S_2(\Gamma_1(fD))$ [15], Theorem 6.5.4; thus one gets h holomorphic forms $\{\phi_1^0, \ldots, \phi_h^0\}$ on the Riemann surface $X_1^0(fD)$. Let $\{A_{\phi_1^0}, \ldots, A_{\phi_h^0}\}$ be the corresponding stationary AF-algebras; the $A_{\phi_i^0}$ are *companion* AF-algebras, see Section 1.2. Recall that the characteristic polynomial for the partial multiplicity matrices $B_{\phi_i^0}$ of companion AF-algebras $A_{\phi_i^0}$ is the same; it is a minimal polynomial of degree h and let $\{\lambda_1, \ldots, \lambda_h\}$ be the roots of such a polynomial, compare with studies of Effros [14], Corollary 6.3. Since $\det(B_{\phi_i^0}) = 1$, the numbers λ_i are algebraic units of the field $\mathcal{K}_\varphi$. Moreover, λ_i are algebraically conjugate and can be taken for generators of the extension $\mathcal{K}_\phi \mid \mathbb{Q}$; since $\deg(\mathcal{K}_\phi \mid \mathbb{Q}) = h = |Cl(\mathfrak{R}_\mathfrak{f})|$ there exists a natural action of group $Cl(\mathfrak{R}_\mathfrak{f})$ on these generators. The action extends to automorphisms of the entire field $\mathcal{K}_\varphi$ preserving $\mathbb{Q}$; thus one gets the Galois group of extension $\mathcal{K}_\phi \mid \mathbb{Q}$ and an isomorphism $Gal\,(\mathcal{K}_\phi \mid \mathbb{Q}) \cong Cl\,(\mathfrak{R}_\mathfrak{f})$. Let us pass to a step-by-step argument.

(i) Let $h := g(X_1^0(fD)) = |Cl\,(R_f)| = |Cl\,(\mathfrak{R}_\mathfrak{f})|$ and let $\phi(z) \in S_2(\Gamma_1(fD))$ be the Hecke eigenform. It is known that there exists $\{\phi_1, \dots, \phi_h\}$ conjugate Hecke eigenforms, so that $\phi\,(z)$ is one of them [15], Theorem 6.5.4. Let $\{\phi_1^0, \dots, \phi_h^0\}$ be the corresponding forms on the Riemann surface $X_1^0(fD)$.

Remark 4. The forms $\{\phi_1^0, \dots, \phi_h^0\}$ can be taken for a basis in the space $\Omega_{hol}(X_1^0(f\,D))$; we leave it to the reader to verify, that abelian variety A_φ is isomorphic to the quotient of $\mathbb{C}^h$ by the lattice of periods of holomorphic differentials $\phi_i^0(z)dz$ on $X_1^0(fD)$.

(ii) Let $\mathbb{A}_{\phi_i^0}$ be the AF-algebra corresponding to holomorphic differential $\phi_i^0(z)dz$ on $X_1^0(fD)$, see Section 2.2; the set $\{\mathbb{A}_{\phi_1^0}, \dots, \mathbb{A}_{\phi_h^0}\}$ consists of the companion AF-algebras. It is known that each $\mathbb{A}_{\phi_i^0}$ is a stationary AF-algebra, i.e. its partial multiplicity matrix is a constant; we shall denote such a matrix by $B_{\phi_i^0}$.

(iii) By definition, the matrices $B_{\phi_i^0}$ of companion *AF*-algebras $\mathbb{A}_{\phi_i}$ have the same characteristic polynomial $p(x) \in \mathbb{Z}[x]$; the matrices $B_{\phi_i^0}$ itself are not pairwise similar and, therefore, the AF-algebras $\mathbb{A}_{\phi_i^0}$ are not pairwise isomorphic. The total number h of such matrices is equal to the class number of the endomorphism ring of pseudo-lattice $K_0^+(\mathbb{A}_{\phi_i^0}) \cong \mathbb{Z} + \mathbb{Z}\theta_1^i + \dots + \mathbb{Z}\theta_{h-1}^i \subset \mathcal{K}_\phi$, [14].

Remark 5. Notice that there are $\{X_1, \dots, X_h\}$ pairwise non-isomorphic Riemann surfaces $X := X_1^0(fD)$ endowed with a holomorphic map $X_i \to \varepsilon_i$ where, $\{\varepsilon_1, \dots, {}_h\}$ are pairwise non-isomorphic elliptic curves $\mathcal{E}_{CM}^{(-D,f)}$ corresponding to elements of the group $Cl(R_f)$. Thus the companion AF-algebras $\{\mathbb{A}_{\phi_1^0}, \dots, \mathbb{A}_{\phi_h^0}\}$ can be viewed as coordinate rings of $\{X_1, \dots, X_h\}$ the latter means that $\mathbb{A}_{\phi_i^0}$ discern non-isomorphic Riemann surfaces and $K_0^+(\mathbb{A}_{\phi_i^0}) \cong \mathbb{Z} + \mathbb{Z}\theta_1^i + \dots + \mathbb{Z}\theta_{h-1}^i \subset \mathcal{K}_\phi$ represents the moduli space of $X_1^0(fD)$.

(iv) The polynomial $p(x)$ is minimal and splits in the totally real field $\mathcal{K}_\varphi$. Indeed, matrices $^{S_2(\Gamma_1(N}$ generate the Hecke algebra $\mathbb{T}_N$ on $S_2(\Gamma_1(N))$; thus each $B_{\phi_i^0}$ is self-adjoint and, therefore, all eigenvalues are real of multiplicity one; since $B_{\phi_i^0}$ is integer, all roots of characteristic polynomial $p(x)$ of $B_{\phi_i^0}$ belong to the field $\mathcal{K}_\varphi$.

(v) Let $p(x) = (x - \lambda_1)\dots(x - \lambda_h)$. It is easy to see that λ_i are algebraic units of the field $\mathcal{K}_\varphi$ because $\det(B_{\phi_i^0}) = 1$; note that numbers $\{\lambda_1, \dots, \lambda_h\}$ are algebraically conjugate. Since $\deg(\mathcal{K}_\phi \mid \mathbb{Q}) = h$, the numbers λ_i can be taken for generators of the field $\mathcal{K}_\varphi$, i.e. $\mathcal{K}_\phi = \mathbb{Q}(\lambda_1, \dots, \lambda_h)$.

(vi) Finally, let us establish an explicit formula for the isomorphism

$$Cl\,(\mathfrak{R}_\mathfrak{f}) \to Gal\,(\mathcal{K}_\phi \mid \mathbb{Q}) \tag{12}$$

Since $Gal\,(\mathcal{K}_\phi \mid \mathbb{Q})$ is an automorphism group of the field $\mathcal{K}_\varphi$ preserving $\mathbb{Q}$, it will suffice to define the action of an element $a \in Cl\,(\mathfrak{R}_\mathfrak{f})$ on the generators λ_i of $\mathcal{K}_\varphi$. Let $\{a_1, \dots, a_h\}$ be the set of all elements of the group $Cl\,(\mathfrak{R}_\mathfrak{f})$. For an element $a \in Cl\,(\mathfrak{R}_\mathfrak{f})$ define an index function α by the formula $a_i a = a_{\alpha(i)}$. Then the action $*$ of an element $a \in Cl\,(\mathfrak{R}_\mathfrak{f})$ on the generators λ_i of the field $\mathcal{K}_\varphi$ is given by the formula:

$$a * \lambda_i := \lambda_{a(i)}, \qquad \forall a \in Cl\,(\mathfrak{R}_\mathfrak{f}). \tag{13}$$

It is easy to verify that formula (13) gives an isomorphism $Cl\,(\mathfrak{R}_\mathfrak{f}) \to Gal\,(\mathcal{K}_\phi \mid \mathbb{Q})$, which is independ of the choice of $\{a_i\}$ and $\{\lambda_i\}$. This argument completes the proof of lemma 3.

Remark 6. The class field theory says that $\mathfrak{f} = f^m$, i.e. the extensions of elds k and $\mathfrak{k}$ must ramify over the same set of prime ideals. Indeed, consider the commutative diagram below, where I_f and $I_\mathfrak{f}$ are groups of all ideals of k and $\mathfrak{k}$, which are relatively prime to the principal ideals (f) and $(\mathfrak{f})$, respectively. Since $Gal\,(\mathcal{K}^{ab}(k) \mid \mathbb{Q}) \cong Gal\,(\mathcal{K}_\phi \mid \mathbb{Q})$ one gets an isomorphism $I_f \cong I_\mathfrak{f}$, i.e. $\mathfrak{f} = f^m$ for some positive integer m.

$$\begin{tikzcd}
I_f \arrow[r, "\text{Artin}", "\text{homomorphism}"'] \arrow[d] & Gal\,(\mathcal{K}^{ab}(k) | \mathbb{Q}) \arrow[d] \\
I_\mathfrak{f} \arrow[r, "\text{Artin}", "\text{homomorphism}"'] & Gal\,(\mathcal{K}_\phi | \mathbb{Q})
\end{tikzcd}$$

Corollary 3. *The Hilbert class field of real quadratic field* $\mathfrak{k} = Q(\sqrt{D})$ *modulo conductor* $\mathfrak{f} \geq 1$ *is isomorphic to the field* $\mathfrak{k}(\mathcal{K}_\varphi)$ *generated by the Fourier coefficients of the Hecke eigenform* $\phi(z) \in S_2(\Gamma_1(fD))$.

Proof. As in the classical case of imaginary quadratic fields, notice that $\deg(\mathcal{K}_\phi \mid \mathbb{Q}) = \deg(\mathfrak{k}(\mathcal{K}_\phi) \mid \mathfrak{k}) = Cl\,(\mathfrak{R}_\mathfrak{f})$; therefore corollary 3 is an implication of lemma 3 and isomorphism $Gal\,(\mathcal{K}_\phi \mid \mathbb{Q}) = Gal\,(\mathfrak{k}(\mathcal{K}_\phi) \mid \mathfrak{k}) = Cl\,(\mathfrak{R}_\mathfrak{f})$.

Theorem 1 follows from corollary 3.

Examples

Along with the method of Stark's units [19], theorem 1 can be used in the computational number theory. For the sake of clarity, we shall consider the simplest examples; the rest can be found in Table 1.

Example 1. Let D = 15. The class number of quadratic field $k = \mathbb{Q}(\sqrt{-15})$ is known to be 2; such a number for quadratic field $\mathfrak{k} = \mathbb{Q}(\sqrt{15})$ is also equal to 2. Thus

$$Cl\,(\mathfrak{R}_{\mathfrak{f}=1}) \cong Cl\,(R_{f=1}) \cong \mathbb{Z}/2\mathbb{Z}, \tag{14}$$

and isomorphism (2) is trivially satised for each power m, i.e. one obtains an unramied extension. By theorem 1, the Hilbert class field of k is generated by the Fourier coefficients of the Hecke eigenform $\phi(z) \in S_2(\Gamma_1(15))$. Using the computer programme *SAGE* created by William A. Stein, one finds an irreducible factor $p(x) = x^2 - 4x + 5$ of the characteristic polynomial of the Hecke operator $T_{p=2}$ acting on the space $S_2(\Gamma_1(15))$. Therefore, the Fourier coefficient $c(2)$ coincides with a root of equation $p(x) = 0$; in other words, we arrive at an extension of k by the polynomial $p(x)$. The generator x of the field $\mathcal{K}_\varphi = \mathbb{Q}(c(2))$ is a root of the bi-quadratic equation $[(x2)^2 + 1]^2 - 15 = 0$; it is easy to see that $x = 2 + \sqrt{-1 + \sqrt{15}}$. One concludes, that the field $\mathcal{K}_\phi \cong \mathbb{Q}\sqrt{-1 + \sqrt{15}}$ is the Hilbert class field of quadratic field $\mathfrak{k} = \mathbb{Q}\sqrt{15}$.

Example 2. Let D = 14. It is known, that for the quadratic field $k = \mathbb{Q}(\sqrt{-14})$ we have $|Cl\,(R_{f=1})| = 4$, while for the quadratic field $Cl\,(\mathfrak{R}_{\mathfrak{f}=1}) = 1$. it holds $Cl\,(\mathfrak{R}_{\mathfrak{f}=1}) = 1$. However, for the ramified extensions one obtains the following isomorphism:

$$Cl\,(\mathfrak{R}_{\mathfrak{f}=2^3}) \cong Cl\,(R_{f=2}) \cong \mathbb{Z}/4\mathbb{Z}, \tag{15}$$

where m = 3 is the smallest integer satisfying formula (2). By theorem 1, the Hilbert class field of $\mathfrak{k}$ modulo $\mathfrak{f}$ = 8 is generated by the Fourier coefficients of the Hecke eigenform $\varphi(z) \in S_2(\Gamma_1(2 \times 4))$. Using the *SAGE*, one finds that the characteristic polynomial of the Hecke operator $T_{p=3}$ on $S_2(\Gamma_1(2 \times 4))$ has an irreducible factor $p(x) = x^4 + 3x^2 + 9$. Thus the Fourier coecient $c(3)$ is a root of the polynomial $p(x)$ and one gets an extension of $\mathfrak{k}$ by the polynomial $p(x)$. In other words, generator x of the field $\mathcal{K}_\varphi = \mathbb{Q}(c(3))$ is a root of the polynomial equation $(x^4 +$

D	f	$Cl\,(R_f)$	f	Hilbert class field of $\mathbb{Q}(\sqrt{D})$ modulo conductor f
2	1	trivial	1	$\mathbb{Q}(\sqrt{2})$
3	1	trivial	1	$\mathbb{Q}(\sqrt{3})$
7	1	trivial	1	$\mathbb{Q}(\sqrt{7})$
11	1	trivial	1	$\mathbb{Q}(\sqrt{11})$
14	2	$\mathbb{Z}/4\mathbb{Z}$	8	$\mathbb{Q}\left(\sqrt[4]{-27+8\sqrt{14}}\right)$
15	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Q}\left(\sqrt{-1+\sqrt{15}}\right)$
19	1	trivial	1	$\mathbb{Q}(\sqrt{19})$
21	2	$\mathbb{Z}/4\mathbb{Z}$	8	$\mathbb{Q}\left(\sqrt[4]{-3+2\sqrt{21}}\right)$
35	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Q}\left(\sqrt{17+\sqrt{35}}\right)$
43	1	trivial	1	$\mathbb{Q}(\sqrt{43})$
51	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Q}\left(\sqrt{17+\sqrt{51}}\right)$
58	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Q}\left(\sqrt{-1+\sqrt{58}}\right)$
67	1	trivial	1	$\mathbb{Q}(\sqrt{67})$
82	1	$x^4-2x^3+4x^2-8x+16$	1	$x^4-2x^3+4x^2-8x+16$
91	1	$\mathbb{Z}/2\mathbb{Z}$	1	$\mathbb{Q}\left(\sqrt{-3+\sqrt{91}}\right)$

Table 1: Square-free discriminants 2 ≤ D ≤ 101.

$3x^2+9)^2-4\times 14=0$. The bi-quadratic equation $x^4+3x^2+9-2\sqrt{14}=0$ has discriminant $-27+8\sqrt{14}$ and one finds a generator of $\mathcal{K}_\varphi$ to be $\sqrt[4]{-27+8\sqrt{14}}$. Thus the field $\mathbb{Q}\sqrt[4]{-27+8\sqrt{14}}$ is the Hilbert class over $\mathbb{Q}(\sqrt{14})$ modulo conductor $\mathfrak{f}=8$. Clearly, the extension is ramified over the prime ideal $\mathfrak{p}=(2)$.

Remark 7. Table 1 above lists quadratic fields for some square-free discriminants $2\le D\le 101$. The conductors f and f satisfying equation (2) were calculated using tables for the class number of non-maximal orders in quadratic fields posted at www.numbertheory.org; the site is maintained by Keith Matthews. We focused on small conductors; the interested reader can compute the higher conductors using a pocket calculator. In contrast, computation of generator x of the Hilbert class field require the online program *SAGE* created by William A. Stein. We write an explicit formula for x or its minimal polynomial $p(x)$ over $\mathfrak{k}$.

Acknowledgment

I thank Yu. I. Manin for helpful correspondence.

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Citation: Nikolaev IV (2016) Real Multiplication Revisited. J Generalized Lie Theory Appl S2: 007. doi:10.4172/1736-4337.S2-007

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, Unversity of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications
ISSN: 1736-4337

Anona, J Generalized Lie Theory Appl 2016, S2
http://dx.doi.org/10.4172/1736-4337.S2-008

Research Article Open Access

Mathematical Aspects of Sikidy

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Abstract

It emphasizes the mathematical aspects of the formation of sikidy. The sikidy as an art of divination is transmitted by oral tradition, the knowledge of these mathematical relationships allows for a more consistent language of sikidy. In particular, one can calculate systematically all ”into sikidy” special tables of Sikidy used in the ”ody” (kind of talismans).

Keywords: Sikidy; Divination; Into; Abelian groups

Introduction

Divination held a special place for all peoples and all times. In Madagascar, the sikidy is an enough precise art of divination and a remedy to avert the fate of the consultant. The present paper is the English version of an unpublished paper [1] translated by Randriambololondrantomalala. Only, introduction and bibliography of the original paper are modified. I'm a differential geometer and Lie theory specialist in this previous domain as [2-6] show. But, this Malagasy art of divination which has mathematical practices has fascinated me and paid my intention in quality of mathematician. So, I made this paper as a first step of long studies about the powerful of mathematics in another fields like divinations. The original paper had plenary lecture in a scientific conference at the University of Fianarantsoa, Madagascar. Next, I think that publication of my paper [1] will be useful as well as several authors have cited my results cf. [7-11]. This article is the first step of my research in this area and mainly, my motivation is to build an algorithm about Sikidy's practice in my next paper.

Formation of Sikidy and Mathematical Relations

Generally, we use seeds of ”kily” (Tamarind), the total number of seeds must be even and large enough to make all desired combinations, at least a hundred seeds. We awakens the sikidy by an invocation that expresses a certain oral tradition of Sikidy and, formulates at the end the questions that we want to have the answers, while turning in circles and always with the right hand the seeds of kily on a mat. Then we take a handful of seeds from the pile at random. It would be at this level the intervention of the Hereafter. We compute the seeds in pairs, if the handle is even number, we align two seeds; in the odd case, we align one seed.

This forms a table from right to left, said mother-sikidy:

$$\begin{matrix} a_{14} & a_{13} & a_{12} & a_{11} \\ a_{24} & a_{23} & a_{22} & a_{21} \\ a_{34} & a_{33} & a_{32} & a_{31} \\ a_{44} & a_{43} & a_{42} & a_{41} \end{matrix}$$

Each variable a_{ij} is composed of one seed or two depending on the result obtained by the above method. The index *i* indicates the position of the line, *j* that of the column ranging from 1 to 4. The quadruplet $(a_{11}, a_{21}, a_{31}, a_{41})$ designates Tale (Consultant); next $(a_{12}, a_{22}, a_{32}, a_{42})$ Maly (Wealth), $(a_{13}, a_{23}, a_{33}, a_{43})$ Fahatelo (A third person); next one $(a_{14}, a_{24}, a_{34}, a_{44})$ Blady (Earth); $(a_{11}, a_{12}, a_{13}, a_{14})$ Fianahana (Child); $(a_{21}, a_{22}, a_{23}, a_{24})$ Abily (An elderly person); next $(a_{31}, a_{32}, a_{33}, a_{34})$ Alisay (Woman); $(a_{41}, a_{42}, a_{43}, a_{44})$ Fahavalo (Enemy).

We build eight other figures below the mother-sikidy left to right derived from the above quadruplets respecting the following law:

- One seed and one seed yield two seeds,
- Two seeds and one seed give one seed,
- Two seeds and two seeds yield two seeds.

So the law of inner composition of Abelian group $(\mathbb{Z}/2\mathbb{Z}, +)$; two seeds represent the identity element $\dot{0}$, one seed is $\dot{1}$. The combining operation is done in $(\mathbb{Z}/2\mathbb{Z})^4$ (quadruplets of $\mathbb{Z}/2\mathbb{Z}$). Thus, we obtain:

- Fahasivy (9, ninth or talisman) = Alisay (7) + Fahavalo (8);
- Haja (11, honor or food) = Fianahana (5) + Abily (6);
- Asorita (13, spirits of deads, or authorities) = Fahatelo (3) + Blady (4);
- Lalana (15 Road) = Tale (1) + Maly (2).

Then combined the above results to have:

- ombiasa (10, soothsayer) Fahasivy = (9) + Haja (11);
- Sely (14 people) = Asorita (13) + Lalana (15);
- Aky (12, god) = ombiasa (10) + Sely (14).

The last figure is:

- Kiba (16, house) = Aky (12) + Tale (1).

Then, an array of sikidy is written:

$$\begin{matrix} (4) & (3) & (2) & (1) & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ a_{14} & a_{13} & a_{12} & a_{11} & \leftarrow (5) \\ a_{24} & a_{23} & a_{22} & a_{21} & \leftarrow (6) \\ a_{34} & a_{33} & a_{32} & a_{31} & \leftarrow (7) \\ a_{44} & a_{43} & a_{42} & a_{41} & \leftarrow (8) \end{matrix}$$

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Received January 29, 2016; **Accepted** March 02, 2016; **Published** March 04, 2016

Citation: Anona FM (2016) Mathematical Aspects of Sikidy. J Generalized Lie Theory Appl S2: 008. doi:10.4172/1736-4337.S2-008

Citation: Anona FM (2016) Mathematical Aspects of Sikidy. J Generalized Lie Theory Appl S2: 008. doi:10.4172/1736-4337.S2-008

$$\begin{array}{cccccccc}
(9) & (10) & (11) & (12) & (13) & (14) & (15) & (16) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
a_{31}+a_{41} & \sum\limits_{1\le i\le 4} a_{i1} & a_{11}+a_{21} & \sum\limits_{1\le i,j\le 4}\left(a_{i1}+a_{1j}\right) & a_{13}+a_{14} & \sum\limits_{1\le j\le 4} a_{1j} & a_{11}+a_{12} & \sum\limits_{1\le i,j\le 4}\left(a_{i1}+a_{1j}\right)+a_{11} \\
a_{32}+a_{42} & \sum\limits_{1\le i\le 4} a_{i2} & a_{12}+a_{22} & \sum\limits_{1\le i,j\le 4}\left(a_{i2}+a_{2j}\right) & a_{23}+a_{24} & \sum\limits_{1\le j\le 4} a_{2j} & a_{21}+a_{22} & \sum\limits_{1\le i,j\le 4}\left(a_{i2}+a_{2j}\right)+a_{21} \\
a_{33}+a_{43} & \sum\limits_{1\le i\le 4} a_{i3} & a_{13}+a_{23} & \sum\limits_{1\le i,j\le 4}\left(a_{i3}+a_{3j}\right) & a_{33}+a_{34} & \sum\limits_{1\le j\le 4} a_{3j} & a_{31}+a_{32} & \sum\limits_{1\le i,j\le 4}\left(a_{i3}+a_{3j}\right)+a_{31} \\
a_{34}+a_{44} & \sum\limits_{1\le i\le 4} a_{i4} & a_{14}+a_{24} & \sum\limits_{1\le i,j\le 4}\left(a_{i4}+a_{4j}\right) & a_{43}+a_{44} & \sum\limits_{1\le j\le 4} a_{4j} & a_{41}+a_{42} & \sum\limits_{1\le i,j\le 4}\left(a_{i4}+a_{4j}\right)+a_{41}
\end{array}$$

As a result, the number of sikidy's tables is determined by the mother-sikidy or $2^{16} = 65536$.

There are two categories of sikidy's figures, princes whose number of seeds is even, and slaves to the odd number of seeds.

The tradition imposes this rule:

"A sikidy can not be interpreted if the Aky (12) is not a prince."

In fact, if we sum all elements of Aky, we have

$$2\sum_{1\le i,j\le 4} a_{ij}$$

Even if the Aky couldn't be a prince, we were wrong calculation.

For clarification on the interpretation, we continue to combine the figures that appear on the table [12]. Thereupon, we must take into account the mathematical links of the sikidy, otherwise we may give different meanings for the same thing [12]:

"The ninth (9) and the healer (10) give the leaves or plants to be used as medicines, ravin'ody".

In fact, given the binary operation of the group, the combination of Fahasivy (9) and Ombiasa (10) gives Haja (11), that is to say a figure that already exists on the table (the relationship (9) + (11) = (10) is equivalent to (9) + (10) = (11)).

There are other examples of contradictions.

Into Sikidy

The sixteen figures of Sikidy are classified according to the cardinal directions. The classification below is used mainly in the southern region of Madagascar. Subscripted letters above each figure will be used to identify the respective figures.

1. Group of the east:

E_1	E_2	E_3
••	••	••
••	•	•
•	•	••
•	••	••
Adabara	Alatsimay	Alamora

2. Group of the north:

N_1	N_2	N_3	N_4
•	•	•	••
••	•	••	••
•	•	••	•
••	••	••	••
Adalo	Karija	Alimizaha	Alibiavo

3. Group of the West:

O_1	O_2	O_3	O_4	O_5
••	••	•	••	•
•	•	••	••	••
••	•	••	••	•
•	•	•	•	•
Alohotsy	Alakaosy	Alokola	Alikisy	Alikarabo

4. Group of the South:

S_1	S_2	S_3	S_4
••	•	•	•
••	•	•	•
••	•	••	••
••	•	••	•
Asombola	Tareky	Asoralahy or Alasady	Betsivongo or Alikasazy

Naturally, these figures have specific meanings [12].

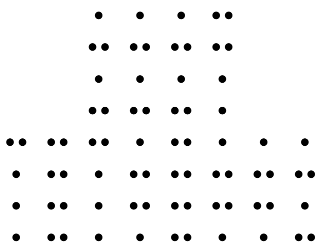
If we denote by P_k, $k = 1, \ldots, 16$, Tale's locations (1) at Kiba (16) in an array of sikidy; we call "Into" the case where one and only one representative of a group appears only once on sixteen seats, P_1 to P_{16}.

Example: Adabara "Into" to the Tale.

+	E1	E2	E3	N1	N2	N3	N4	O1	O2	O3	O4	O5	S1	S2	S3	S4
E1	S1	O1	O2	O3	S4	O5	O4	E2	E3	N1	N4	N3	E1	S3	S2	N2
E2	O1	S1	N4	S3	N3	N2	E3	E1	O4	S2	O2	S4	E2	O3	N1	O5
E3	O2	N4	S1	N2	N1	S3	E2	O4	E1	S4	O1	S2	E3	O5	N3	O3
N1	O3	S3	N2	S1	E3	N4	N3	S2	S4	E1	O5	O4	N1	O1	E2	O2
N2	S4	N3	N1	E3	S1	E2	S3	O5	O3	O2	S2	O1	N2	O4	N4	E1
N3	O5	N2	S3	N4	E2	S1	N1	S4	S2	O4	O3	E1	N3	O2	E3	O1
N4	O4	E3	E2	N3	S3	N1	S1	O2	O1	O5	E1	O3	N4	S4	N2	S2
O1	E2	E1	O4	S2	O5	S4	O2	S1	N4	S3	E3	N2	O1	N1	O3	N3
O2	E3	O4	E1	S4	O3	S2	O1	N4	S1	N2	E2	S3	O2	N3	O5	N1
O3	N1	S2	S4	E1	O2	O4	O5	S3	N2	S1	N3	N4	O3	E2	O1	E3
O4	N4	O2	O1	O5	S2	O3	E1	E3	E2	N3	S1	N1	O4	N2	S4	S3
O5	N3	S4	S2	O4	O1	E1	O3	N2	S3	N4	N1	S1	O5	E3	O2	E2
S1	E1	E2	E3	N1	N2	N3	N4	O1	O2	O3	O4	O5	S1	S2	S3	S4
S2	S3	O3	O5	O1	O4	O2	S4	N1	N3	E2	N2	E3	S2	S1	E1	N4
S3	S2	N1	N3	E2	N4	E3	N2	O3	O5	O1	S4	O2	S3	E1	S1	O4
S4	N2	O5	O3	O2	E1	O1	S2	N3	N1	E3	S3	E2	S4	N4	O4	S1

Table 1: Inner law of composition.

Citation: Anona FM (2016) Mathematical Aspects of Sikidy. J Generalized Lie Theory Appl S2: 008. doi:10.4172/1736-4337.S2-008



In this example, Adabara, E_1 one member of the group of the East is located at P_1, the other E_l, $l = 1,2,3$ don't take place at P_k, $2 \leq k \leq 16$.

In that table of Sikidy, we say that the Sikidy gives a formal advice (mitoka vava). The consultant would be successful.

To calculate these "Into", we use the following Table 1 of inner law of composition:

Compute these "Into" using this table is elementary. The total of the "Into" for one figure in the place P_k takes its value from 0 to a hundred. For example, Alohotsy "Into" to Ombiasa (who means the Great Divine), searched by the Mpisikidy, doesn't exist, but Alohotsy "Into" to Sely which has total number 8. Adabara "Into" to Tale has 132,...etc.

These "Into" Sikidy have particular significations and truthfully considered. We use them to get talismans.

Conclusion

In the present paper, Anona investigated mainly the mother-Sikidy. In a next paper in the same topic, he will make the Sikidy more precise in order to check daughter-Sikidy and so on. The language of the Sikidy is very large. Frequently, the data obtained from the Mpisikidy are contradictory. Certainly, make a coherent language of the Sikidy throughout different collects of data is very interesting.

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Citation: Anona FM (2016) Mathematical Aspects of Sikidy. J Generalized Lie Theory Appl S2: 008. doi:10.4172/1736-4337.S2-008

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, Unversity of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications
ISSN: 1736-4337

Lantonirina, J Generalized Lie Theory Appl 2016, S2
http://dx.doi.org/10.4172/1736-4337.S2-009

Research Article Open Access

Trying to Explicit Proofs of Some Vey's Theorems in Linear Connections

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Abstract

Let X a diferentiable paracompact manifold. Under the hypothesis of a linear connection r with free torsion T on X, we are going to give more explicit the proofs done by Vey for constructing a Riemannian structure. We proposed three ways to reach our object. First, we give a sufficient and necessary condition on all of holonomy groups of the connection ∇ to obtain Riemannian structure. Next, in the analytic case of X, the existence of a quadratic positive definite form g on the tangent bundle TX such that it was invariant in the infinitesimal sense by the linear operators $\nabla^k R$, where R is the curvature of ∇, shows that the connection ∇ comes from a Riemannian structure. At last, for a simply connected manifold X, we give some conditions on the linear envelope of the curvature R to have a Riemannian structure.

Keywords: Linear connections; Riemannian connection; Levi-civita connection; Holonomy groups; Linear en-velope; K^{th}derivations; Lie algebras

Preliminary and Introduction

In 1978 Vey was the examinator of Anona's phd at Institut Fourier Grenoble. In this time, Vey was written some theorems in linear connections. The title of this unpublished paper was: "Sur les connexions riemanniennes" means "On the Riemannian connections". They had discussions, as said Vey, this result was not well explicit. So, one of the motivations of this paper is to explicit some of them about linear connections. First, we consider a paracompact manifoldX, we are looking for a Riemannian metric g which is invariant by parallel transport such that it produces a linear connection∇ with ∇g = 0. Next, we assume that X is a simply connected real analytic manifold accompanied by a real analytic connection∇, these new assumptions construct us a positive definite quadratic form g (satisfying ∇g = 0) infinitesimally preserved by the infinitesimal holonomy group. This is obtaining by the fact that the Lie algebra of holonomy groups coincides with the Lie algebra of the infinitesimal holonomy group. Finally, we present our problem so as to consider the linear envelope of the curvature, in the case where X is a smooth manifold. However, under these conditions that the linear envelope of the curvature of constant dimension coincides with the Lie algebra of the holonomy group, make the quadratic form g parallel to∇. Thus builds a Riemannian structure. Recently, several authors treat the same questions as R. Feres in and A. Vanzurava [1, 2]. This last author gave an algorithm for constructing Riemannian structure which is similar as Veydone in the end of his paper. This redaction gives an interest for a next one where we use some results and idea of the present paper. So, let us recall some useful definitions in linear connections.

Definition 1

First, let X is a smooth manifold,$T_x X$, $x \in X$ the vector space of tangent vectors on x and T X the tangent bundle defined by $T = \bigcup_{x\in X} T_x X$. Recall that a Riemannian manifold X is a smooth manifold equipped with a Riemannian structure g on X which is defined by the morphism of bundles

$$g : TX \times TX \to \mathbb{R}$$

such that $\forall x \in X$

$$g_{x:x} X \times T_x X \to \mathbb{R}$$

defined an inner product with $\forall \quad U,V \in TX$

$$X \to \mathbb{R}$$

$$x \mapsto g_x(U(x),V(x))$$

is differentiable.

Definition 2

Let X be a smooth manifold, let $\Gamma(X)$ be the module of vector fields on X and $F(X) = \{ f : X \to \mathbb{R} \;\; smooth \}$ the ring of real smooth functions. We recall that a linear connection on X is the application

$$: \Gamma(X) \times \Gamma(X) \to \Gamma(X)$$

$$(U, V)\ \nabla_U V$$

such that, $\forall \; U, U', V, V' \in \Gamma(X), f \in F(X)$ we have:

i. $\nabla_{U+U'}V = \nabla_U V + \nabla_{U'} V$

ii. $\nabla_{fU} V = f\nabla_U V$

iii. $\nabla_U(V+V') = \nabla_U V + \nabla_U V'$

iv. $\nabla_U(fV) = f\nabla_U V + U(f)V$

Theorem 3

(Fundamental Theorem of Riemannian Geometry) Let X be a Riemannian manifold. Any Riemannian structure g produces an unique linear connection ∇ called Levi-Civita's connection on X with free torsion such that $\nabla g=0$.

The Holonomy Groups and Riemannian Structure

Definition 4

Let X be a smooth manifold, a path is a smooth function from [0, 1] to X.

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Received January 23, 2016; **Accepted** February 15, 2016; **Published** March 15, 2016

Citation: Lantonirina LS (2016) Trying to Explicit Proofs of Some Vey's Theorems in Linear Connections. J Generalized Lie Theory Appl S2: 009. doi:10.4172/1736-4337.S2-009

Definition 5

let X be a smooth manifold, x a point of X [3]. We call holonomy groups based on a point x of

the Levi-Civita's connection ∇, or Riemannian holonomy, denoted by $Hol(x)$ the set

$$Hol(x)=\{\tau^{\gamma}: T_xX \to T_xX\}$$

where $\gamma : [0, 1] \to X$ is closed path along x to x such that $\gamma(0)=\gamma(1)=x$

Proposition 6

Let X be a smooth manifold, and $x, y \in X$ [3]. If X is connected and γ a path x to y, the

holonomy groups of the point x, y are isomorphic and we have

$$\tau^{\gamma} Hol(x) \tau^{\gamma^{-1}} = Hol(y).$$

Definition 7

Let K be a compact group. We call that an Haar measure on K denoted μ is the unique invariant measure by right and left translation in K, then

$$\int_K f(k)\,d\mu(k) = \int_K f(kh)\,d\mu(k) = \int_K f(hk)\,d\mu(k),$$

such that $f \in L(K)$ a ring of all functions $f: K \to \mathbb{C}$.

Definition 8

Let X be a Riemannian manifold, g_0 a metric on X, K a compact group. We define an invariant metric g on X which normalizes any metric g_0 of X on the compact group K. Let $g=\int_K k^* g_0 dk$ Where k^* is the "pull-back "of k.

This integration follows the Haar measure property.

Proposition 9

Recall that all paracompact manifolds X admit a Riemannian metric [4].

Proposition 10

Let X be a smooth manifold and the linear connection on X with free torsion. ∇ proceeds from a Riemannian structure if and only if its holonomy groups are relatively compact.

Proof.

On the one hand, suppose ∇ comes from a Riemannian structure (X, g).Then by the Fundamental Theorem of Riemannian Geometry, we have the Levi-Civita's connection ∇ on X. Consequently, the Riemannian holonomy is well defined by the holonomy group of Levi-Civita's connection on X. But, the holonomy group is a subgroup of the orthogonal group $O(n)$ which is compact, so this subgroup is relatively compact.

On the other hand, let us suppose that the holonomy group on each point of $x \in X$ is relatively compact. Let g_0 be a Riemannian metric on the paracompact manifold X, and $x \in X$. Then, it leaves invariant a positive definite quadratic form g on a point x of X on TX. Indeed, we have

$$Hol(x)=\{\tau^{\gamma}; \gamma:[0,1]\to X;\ \gamma(0)=\gamma(1)=x\},$$

then we define,

$$g(U_x,V_x)=\int_{Hol(x)}(\tau^{\gamma})^* g_0(U_x,V_x)\,d\tau^{\gamma};\quad U_x,V_x \in T_xX.$$

Let $,\upsilon^{\alpha} \in Hol(x)$,then we have,

$$\begin{aligned} g(\upsilon^{\alpha}U_x,\upsilon^{\alpha}V_x) &= \int_{Hol(x)}(\tau^{\gamma})^* g_0(\upsilon^{\alpha}U_x,\upsilon^{\alpha}V_x)\,d\tau^{\gamma} \\ &= \int_{Hol(x)} g_0\left(\tau^{\gamma}(\upsilon^{\alpha}U_x),\tau^{\gamma}(\upsilon^{\alpha}V_x)\right)d\tau^{\gamma} \\ &= \int_{Hol(x)} g_0(\tau^{\gamma}U_x,\tau^{\gamma}V_x)\,d\tau^{\gamma} \\ &= \int_{Hol(x)}(\tau^{\gamma})^* g_0(U_x,V_x)\,d\tau^{\gamma} \\ &= g(U_x, V_x) \end{aligned}$$

Let $y \in V_x \subset X$ (V_x neighbourhood of x). Since X a smooth manifold, then X is locally homeomorphic to an open set of $\mathbb{R}^n$. Otherwise, $\mathbb{R}^n$ is locally connected, and locally connected is preserved by homeomorphism. Consequently, under the Proposition 6, the holonomy is independent of the selected base point. Then, the parallel transport relatively to ∇ of the quadratic form g along the path from x to y is independent of the path, by definition of holonomy. Thus, it constructs a Riemannian structure g on X preserved by parallel transport, with $\nabla g = 0$. Since ∇ is of free torsion, and uniqueness of ∇, and then ∇ is a Riemannian connection.

Real analytic manifold and Riemannian structure

For the following, we suppose X a simply connected manifold to have an explicit result.

Definition 11

Let x a point of X, denoted by $C(x)$ the set of closed curves on x, and $C^0(x)$ the set of the contractible curves on point x. Take a point $u \in p^{-1}(x)$, where p is the projection of the linear structure bundle $L(X)$ at X *cf.* [5]. We recall that the holonomy group on x denoted by $\psi(x)$ is the subgroup of the diffeomorphism $p^{-1}(x)$ whose the element are obtained by parallel transport of the curves in $C(x)$, that is to say, the element of $\psi(x)$ are of the form $\tau: p^{-1}(x) \to \psi^0(x), \to \tau \in C(x)$. In the similar ways, we call holonomy group restricted on point x denoted $\psi^0(x)$ the subgroup of diffeomorphism of $p^{-1}(x)$ whose the element are of the form $\tau: p^{-1}(x)$, $\forall \tau \in C^0(x)$.

- For $u \in p^{-1}(x)$, we define holonomy groups on point u by $\psi^0(u)=\{a \in Gl(n,\mathbb{R}), R_a(u)=\tau(u),$ for $\tau \in \psi(x)\} \in Gl(n,\mathbb{R})$.The same, we define the restricted holonomy groups on point u.

Now, let us define the local holonomy group by $\psi^*(u)=\cap\psi^0(u, U_k)$ where $\psi^0(u, U_k)$ is a subset of $\psi^0(u)$ such that $\psi^0(u, U_k) \subset \psi^0(u, U_{k+1})$, with U_k indicates a sequence the neighbourhood of x satisfied $U_{k+1} \in U_k$, for $k \in \mathbb{Z}^+$ and $\bigcap_{k=1,2,\ldots} U_k = x$.

Proposition 12

Let X be a smooth manifold, $L(X)$ a linear structure fiber on X, H a connection on $L(X)$ [6].

Then, all tangent vectors U of $T_uL(X)$ is called vertical (resp. horizontal) when it belong to V_u (resp. H_u), $u \in L(X)$, where V_u (resp. H_u) indicate the vertical (resp. horizontal) subspace of $T_uL(X)$.

Definition 13

Let $m_0(u)$ a subspace of $gl(n,\mathbb{R})$ generated by $\Omega_u(U; V)$(the curvature form on $L(X)$ *cf.* [6] p.152) the horizontal vectors $U; V \in T_u(L(X))$ with $u \in L(X)$. By recurrence on k, $m_k(u)$ defined a subspace of $gl(n.\mathbb{R})$ generated by the elements of $m_{k-1}(u)$ and the element of the form V_1 _ _ _ _ _ $V_k(\Omega(U; V))$, where $U; V; V_1,\ldots, V_k$ are the horizontal

Citation: Lantonirina LS (2016) Trying to Explicit Proofs of Some Vey's Theorems in Linear Connections. J Generalized Lie Theory Appl S2: 009. doi:10.4172/1736-4337.S2-009

vectors. Let $g'(u)$ the union of all the $m_k(u); k = 0,1,2,\ldots$. The set of $g'(u)$ is a sub-algebra of gl(n;ℝ) and the connected Lie subgroup generated by $g'(u)$ is called infinitesimal holonomy groups on point u of $L(X)$. We denote it by $\psi'(u)$.

For a real analytic manifold, denoted by $Hol(x)$ resp. $Hol(x)$, resp. $Hol(x)$ the holonomy groups (resp. the restricted holonomy group, resp. the infinitesimal holonomy groups) on point x in X.

Proposition 14

Let X be a real analytic manifold, the linear real analytic connection [2]. Denoted by

$\underline{h}(x), \underline{h}^*(x)\ \underline{h}'(x)$ the respective Lie algebras of holonomy groups $Hol(x)$, $Hol^*(x)$, $Hol'(x)$. The two groups $Hol^*(x)$, $Hol'(x)$ constitute a Lie subgroups of $Hol(x)$ namely $Hol'(x) \subset Hol^*(x) \subset Hol(x)$ and con-sequently $\underline{h}'(x) \subset \underline{h}^*(x) \subset \underline{h}(x)$. Since X is a real analytic manifold, we have the reverse inclusion for the holonomy groups and the equality between $\underline{h}(x) = \underline{h}'(x)$.

Theorem 15

Let X be a real analytic manifold,∇ the linear real analytic connection associated to X [6].

The Lie algebra $\underline{h}'(x)$ of the infinitesimal holonomy groups $Hol'(x)$ is spanned by the k^{th} covariant derivatives

$\nabla^k R(U,V;U_1,\ldots,U_K)$, where $U,V,U_1,\ldots,U_k \in T_xX$, $0 \le k < \infty$ and $\nabla^0 R(U,V) = R(U,V)$.

Proposition 16

Let $(X;\nabla)$ be a real analytic simply connected, with free torsion, x a point of X, g a

symmetric bilinear quadratic form on T_xX [7]. Then the invariance of g by $Hol(x)$ is characterized by:

$$g(AU,V) + g(U,AV) = 0, \forall A \in \underline{h}(x), U,V \in T_xX, x \in X. \qquad (1)$$

Proposition 17

Let X be a real analytic manifold, simply connected, ∇ an analytic connection with free torsion

on X and R its curvature. Let x be point of X. If there exist a positive definite quadratic form g on T_xX, preserved in the infinitesimal sens by all of linear operators $\nabla^k R(W,U_1,U_2,\ldots,U_k)$ ($k \ge 0, W \in \Lambda^2 T_xX, U_i \in T_xX$), then ∇ proceeds from a Riemannian structure.

Proof.

Let $x \in X$, g a positive definite quadratic form on T_xX, ∇ the analytic connection associated X, R the curvature of the connection ∇. Suppose that g is preserved in the infinitesimal sense by all linear operators $\nabla^k R(W,U_1,U_2,\ldots,U_k)$

of $\underline{h}'(x)$ (according to the Theorem 15), that is to say that we have according to the

Propositions 16 and 14.

$$g(\nabla^k R(W,U_1,U_2,\ldots,U_k)U,V) + g(U,\nabla^k R(W,U_1,U_2,\ldots,U_k)V) = 0 \quad (2)$$

$\forall W \in \Lambda^2 T_xX, U_i \in T_xX$ and $\nabla^k R(W,U_1,U_2,\ldots,U_k)$ (k integer ≥ 0) in $\underline{h}'(x)$: Since $\nabla^k R(W,U_1,U_2,\ldots,U_k) \in \underline{h}'(x)$ and $\underline{h}'(x) = \underline{h}(x) = \underline{r}_x^{(k)}$, then $\nabla^k R(W,U_1,U_2,\ldots,U_k)$ belongs to the linear envelope in the endomorphism of T_xX. Then the equation (2) comes,

$$\nabla^k_{R(W,U_1,U_2,\ldots,U_k)} g(U,V) = 0 \quad \forall U,V \in T_xX, x \in X,$$

therefore $\nabla^k g = 0$ for all $k \ge 0$, it follows that $\nabla g = 0$. Since the torsion is supposed null, then ∇ comes from a Riemannian structure.

Linear envelope and Riemannian structure

Lemma 18

Let E be a real vector space of finite dimension, and I an interval of ℝ, and $V_i(t)$ a sequence of vectors which depend differentially on $t \in I$. Let $L(t)$ the vector space spanned by $V_i(t)$. Suppose that $L(t)$ of constant dimension, and for all i and for all

$$\frac{d}{dt}V_i(t) \in L(t)$$

then $L(t)$ is independent of t.

Proof.

We proceed like the arguments of proof in [8] pp. 943-944.

Theorem 19

(Ambrose-Singer Theorem) The Lie algebra $\underline{hol}(x_0)$ of the holonomy groups $Hol(x_0)$ (Lie sub-algebra of the Lie algebra g) are spanned by the vectors $\Omega_x(U,V)$ for all point x of a principal bundle P *cf.*[1], link with x_0 by a piecewise smooth curve of extremity x and x_0 and U, V the horizontal vectors on x [9].

Proposition 20

Let X be a smooth manifold, and a linear connection of X. If the tensor ∇R takes all values in the linear envelope of the curvature, and if it has a constant dimension, then it coincides with the Lie algebra of the holonomy groups in each point.

Proof.

Let x a point of X. Let us indicate by $\underline{r}$ the linear envelope of the curvature on $x \in X$; and $\underline{h}(x)$ the Lie algebra of the holonomy. Let y a point of X, and $\tau^\gamma$the parallel transport along (on End (TX)), where γ indicates a path joins x and y. According to the Ambrose-Singer theorem, $\underline{h}(y)$ is the linear envelope of subspaces $\tau^\gamma(\underline{r}_x)$. Take then a point $x \in X$, a path (t) is parametrized in [0; 1] and joins x to y. Recall that R is defined by $R: \Lambda^2 TX \to End(TX)$. Let $w_i (1 \le i \le n(n-1)/2)$ a basis of $\wedge^2 T_xX$ and, $w_i(t)$ the basis of $\wedge^2 T_{\gamma(t)}X$ obtained by parallel transport along of $\gamma, \underline{r}$ defined a linear envelope in End $(T_{\gamma(t)}X)$ of the operators $R_{(t)}(w_i(t))$. Now by hypothesis ∇ R takes its values on the linear envelope of the curvature, then $\nabla_{d/dt} R_{\gamma(t)(w_i(t))} \in \underline{r}_{\gamma(t)}$ Let us consider the subspace of End (T_yX)

$$\underline{r}'_t = \tau^{\gamma|[t,1]}\left(\underline{r}_{\gamma(t)}\right)$$

, since the linear envelope is supposed of constant dimension, then $\underline{r}'_t$ has an independent dimension that is to say independent of t, and it defines the linear envelope of the operators

$$R'_{i,t} = \tau^{\gamma|[0,1]}(R_{\gamma(t)}(w_i(t))) \in END(T_{yX})$$

It result from precedent that for all $t \in [0, 1]$;

$$\frac{d}{dt}R'_{i,t} \in \underline{r}'_t. \qquad (3)$$

Since $R'_{i,t}$ depends differentially of t for $t \in [0; 1]$, $\underline{r}'$ has a constant dimension by hypothesis, and that we have the relation (3). Then according to the Lemma18, $\underline{r}'_t$ t is independent of t. Then,

$\underline{r}'_t = \underline{r}'_1 = \underline{r}'_y$. Consequently $\underline{h}(y) = r_{\underline{y}}$.

Theorem 21 (Lie-Palais) [6]

Let X be a compact smooth manifold and we have an action on X by the Lie algebra of finite dimension, then this is a lift of an action of Lie group of finite dimension.

Proposition 22

Let X be a simply connected manifold, ∇ a linear connection on X with free torsion, and R its curvature. Supposed that the free conditions which follow are verified:

(a) the linear envelope of the curvature R has a constant dimension,

(b) the tensor R takes its values in $\underline{r}$,

(c) in all points x of X, there exists a positive definite quadratic form on T_xX infinitesimally preserved by $\underline{r}_x$,then the connection∇ proceeds from a Riemannian structure on X.

Proof.

Let x a point of X, suppose verified the above hypothesis. Since (a) and (b) are true by hypothesis, then according to the Proposition 20, the linear envelope coincides with the Lie algebra of the holonomy group. Consequently, the linear envelope is stable by the Lie bracket and then admits a Lie algebra structure. Now according to the condition (c), there exist a quadratic positive definite form on T_xX preserved infinitesimally by $\underline{r}_x$ It follows that the linear envelope is compact. We deduce there that the Lie algebra of holonomy groups is of finite dimension see condition (a). Since X is locally compact, then according to Lie-Palais theorem, the Lie algebra of the group is the lift of the holonomy group whose the topologic structure compact is preserved. Therefore, the holonomy group is relatively compact. Now the Proposition 10 says that for X be a smooth manifold, ∇ its connection with free torsion, ∇ comes from a Riemanniann structure if and only if its holonomy group are relatively compact. Then we have the result.

Acknowledgments

I express my acknowledgments firstly to Lord. Next, I also grateful acknowledge Pr. M. F. Anona, Dr.P. Randriambololondrantomalala, Dr.H. S. G. Ravelonirina, at the University of Antananarivo. Finally, a special thanks to my family.

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Citation: Lantonirina LS (2016) Trying to Explicit Proofs of Some Vey's Theorems in Linear Connections. J Generalized Lie Theory Appl S2: 009. doi:10.4172/1736-4337.S2-009

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, University of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications

Barna and Kersner, J Generalized Lie Theory Appl 2016, S2
http://dx.doi.org/10.4172/1736-4337.S2-010

Research Article Open Access

Heat Conduction: Hyperbolic Self-similar Shock-waves in Solid Medium

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Abstract

Analytic solutions for cylindrical thermal waves in solid medium are given based on the nonlinear hyperbolic system of heat flux relaxation and energy conservation equations. The Fourier-Cattaneo phenomenological law is generalized where the relaxation time and heat propagation coefficient have a general power law temperature dependence. From such laws one cannot form a second order parabolic or telegraph-type equation.We consider the original non-linear hyperbolic system itself with the self-similar Ansatz for the temperature distribution and for the heat flux. As results continuous.

Keywords: Self-similar solution; Non-linear heat conduction; Shock wave; Cattaneo heat conduction law

Introduction

Analytic solutions for cylindrical thermal waves in solid medium are given based on the nonlinear hyperbolic system of heat flux relaxation and energy conservation equations. The Fourier-Cattaneo phenomenological law is generalized where the relaxation time and heat propagation coefficient have a general power law temperature dependence. From such laws one cannot form a second order parabolic or telegraph-type equation. We consider the original non-linear hyperbolic system itself with the self-similar Ansatz for the temperature distribution and for the heat flux. As results continuous and shock-wave solutions are presented. For physical establishment numerous materials with various temperature dependent heat conduction coefficients are mentioned.

44.90.+c, 02.30.Jr

In contemporary heat transport theory (ever since Maxwell's paper [1]) it is widely accepted in the literature that only for stationary and weakly non-stationary temperature fields the constitutive equation assumes that a temperature gradient ∇T instantaneously produces heat flux q according to the Fourier law.

$$q(x,t)=-\kappa\nabla T(x,t) \tag{1}$$

Combining this equation with the energy conservation law the usual parabolic heat conduction equation is given. Heat conduction mechanisms can be classified via the temperature dependence of the coefficient $\kappa : T^{\nu}$. There are three different cases of thermal conductivity, normal heat conduction which obeys the Fourier law (ν=0), slow (ν>0) and fast heat conduction $-2<\nu<0$.

In plasma physics if the temperature range is between 10^5K and 10^8 K then the coefficient of the heat conductivity κ depends on the temperature and density of the material. It is usually assumed to have a power dependence $\kappa=\kappa_0 T^{\nu} v^{\mu}$ where $v=1/\rho$ is the specific volume the coefficient κ_0 and the exponents ν, μ depend on the heat conduction mechanism [2]. With radiation heat conduction one has $4 \le \nu \le 6$, $2mm1 \le \mu \le 2$; with electron heat conduction and fully ionized plasma $\nu=5/2$, $2mm\mu=0$. For magnetically confined non-neutral plasma the classical heat conduction coefficient is the following [3] $\kappa \approx \frac{c_1}{\sqrt{T}} ln[c_2 T^{3/2}]$. Parabolic thermal wave theory is based on this approach [2,4]. In plasmas heat conduction is strongly coupled to flow properties which we will not consider in the following. The linear parabolic theory predicts infinite speed of propagation which is known as the "paradox of heat conduction" (PHC). The following two theories resolve this contradiction.

However, if the time scale of local temperature variation is very small, Eq. (1) is replaced by

$$q(x,t=\tau)=-\kappa\nabla T(x,t) \tag{2}$$

Where τ is called the thermal relaxation time. This is a thermodynamic property of the materials which was determined experimentally for large number of materials. Although τ turns out to be very small in many instances e.g. is of order of picoseconds for most metals, there are several materials where this is not the case, most notably sand (21 s), H acid (25 s), $NaHCO_3$ (29 s), and biological tissue (1-100 s) [5].

Unlike the Fourier's heat conduction law, this constitutive equation is non-local in time. The desired local character can be restored with the Taylor expansion of q by time which is usually truncated at the first order namely

$$q(x,t)+\tau\frac{\partial q(x,t)}{\partial t}=-\kappa\nabla T(x,t). \tag{3}$$

This is the well-known Cattaneo heat conduction law [6] the second term on the left hand side is known as the "thermal intertia". Combining this constitutive equation with the energy conservation yields the hyperbolic telegraph heat conduction equation where τ and are constants. Hyperbolic equations usually ensure finite propagation velocity. Unfortunately, the usual telegraph equations has no self-similar solutions which would be a desirable physical property. In the work of [7] a non-autonomous telegraph-type heat conduction equation is presented with self-similar non-oscillating compactly supported solutions. A review with a large number of physical models of heat waves can be found in [5,8]. A recent work on the speed of heat waves was published by [9].

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Received February 29, 2016; **Accepted** April 14, 2016; **Published** May 03, 2016

Citation: Barna IF, Kersner R (2016) Heat Conduction: Hyperbolic Self-similar Shock-waves in Solid Medium. J Generalized Lie Theory Appl S2: 010. doi:10.4172/1736-4337.S2-010

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Our starting point is the following

$$q_t = -\frac{q}{\tau} - \frac{\kappa}{\tau} T_r, \tag{4}$$

$$c_0 T_t = -q_r - \frac{q}{r}. \tag{5}$$

The first equation of the system is the generalized Fourier-Cattaneo heat conduction law and the second one is the energy conservation condition for the radial coordinate. The heat flux $q=q(r, t)$ and the temperature $T=T(r, t)$ have radial coordinate and time dependence. The subscripts r and t denote the partial derivatives with respect to the radial coordinate and the time, respectively. (From now on we investigate the radial coordinate of a cylindrical symmetric problem as spatial dependence.) The parameter $c_0=\rho c$ where ρ is the mass density and c is the specific heat. Second order effects such as compressibility are neglected (ρ and c are constants during the process).

In the following we shall suppose that the heat conduction coefficient and the thermal relaxation depend on temperature on the following way:

$$\kappa=\kappa_0 T^{\omega},\ 1cm\tau=\tau_0 T^{-\delta} \tag{6}$$

The κ_0 and τ_0 are real numbers with the proper physical dimensions. Now our dimensionless system reads:

$$q_t=-T^{\varepsilon}q-T^{\varepsilon+\omega}T_r \tag{7}$$

$$T_t = -q_r - \frac{q}{r}. \tag{8}$$

There are various phenomenological heat conduction laws available for all kind of solids, without completeness we mention some well-known examples. For pure metals according to [10] (Page 275 Eq. 27.3) the Wiedemann-Franz law the thermal conductivity is proportional with the electrical conductivity times the temperature $\kappa=\sigma LT$ The proportionality constant L is the so called Lorentz number with the approximate numerical value of $2.44 \times 10^{-8} W\Omega K^{-2}$. For exact numerical data for various metals see [11]. The relaxation time τ is proportional to the heat conduction coefficient divided by the temperature. For metals with impurities the thermal resistivity (inverse of the thermal conductivity) is $\kappa^{-1}=AT^2+BT^{-1}$ where A and B can be obtained from microscopic calculation based on quantum mechanics [10] (Page 297 Eq. 40.11).

A hard-sphere model for dense fluids from [12] derives a relation where the heat flux $q(x,t)=a\nabla T(x,t)+q^2(x,t)$ which certainly means a non-linear heat propagation process. For the heat conduction in nanofluid suspensions [13] derives the $\kappa \approx c/(T_2-T_1)$ law with additional time dependence. Another exotic and very promising new materials are the carbon nanotubes which have exotic heat conduction properties. Small *et al.* [14] performed heat conductivity measurements and found that at low temperatures there are two distinct regimes $\kappa(T)$: $T^{2.5}(T<50K)$ and $\kappa(t)$: $T^2(50<T<150K)$. Beyond this regime there is a deviance from this quadratic temperature dependence and the maximum κ value lies at 320 K. Above this value - at large temperatures - there is a $\kappa(T)$: $1/T$ dependence according to [15]. Additional nanoscale systems (like silicon films, or multiwall carbon nanotubes) have exotic temperature dependent heat conduction coefficients as well, for more see [16]. For encased graphene the heat conduction coefficient is $\kappa : T^{\beta}$ where $1.5<\beta<2$ at low temperature ($T<150K$) [17]. A recent review of thermal properties of graphene and nanostructured carbon materials can be found in [18].

Our model is presented to describe the heat conduction of any kind of solid state without additional restrictions, therefore room or even higher temperature can be considered with large negative ω exponents.

Even from these examples we can see that it has a need to investigate the general heat conduction problem, where the coefficients have general power law dependence.

We look for the solutions of (7,8) in the most general self-similar form.

$$T=t^{-\alpha}f(\eta),\ q=t^{-\delta}g(\eta). \tag{9}$$

For a better transparency in the following we introduce a new variable $\eta = \frac{r}{t^{\beta}}$, where α, β, δ are all real numbers.

The similarity exponents α, δ and β are of primary physical importance since α, δ represents the rate of decay of the magnitude T or q, while β is the rate of spread (or contraction if <0) of the space distribution as time goes on. Self-similar solutions exclude the existence of any single time scale in the investigated system.

We substitute (9) into (7) and (8). It can be checked that

$$\alpha = \frac{1}{\omega+1},\quad 3mm\beta = \frac{1}{2(\omega+1)},\quad 3mm\delta = \frac{2\quad 3}{2(\omega+1)},\quad 3mm\varepsilon = \omega \quad 1. \tag{10}$$

Then we can obtain the shape functions f and g the following ordinary differential equation (ODE) system

$$\delta g + \beta\eta g' = gf^{\omega+1} + f^{2\omega+1}f', \tag{11}$$

$$(\eta g)' = \beta(\eta^2 f)' \tag{12}$$

where prime means derivation with respect to η.

The first lucky moment is that (12) relates f and g in a simple way

$g=\beta\eta f(13)$

if the $\alpha=2\beta$ universality relation is fulfilled.

Note, that we can immediately read how the self-similar solutions of the temperature distribution T and the heat flux q depend on ω

$$T = t^{\frac{-1}{\omega+1}} f\left(\frac{r}{t^{\frac{1}{2(\omega+1)}}}\right),\quad 8mm q = t^{\frac{2\omega+3}{2(\omega+1)}} g\left(\frac{r}{t^{\frac{1}{2(\omega+1)}}}\right). \tag{14}$$

The parameter dependence of the complete heat conduction coefficient and relaxation time can be expressed via ω as well

$$\kappa = \kappa_0 t^{\frac{-\omega}{\omega+1}} f^{\omega}\left(\frac{r}{t^{\frac{1}{2(\omega+1)}}}\right),\quad \tau = \kappa_0 t^{-1} f^{\omega+1}\left(\frac{r}{t^{\frac{1}{2(\omega+1)}}}\right). \tag{15}$$

Recall that >-1. These are already very informative and useful relations to investigate the global properties of the solutions, note that such kind of analysis are available for large number of complex mechanical and flow problems [19].

Substituting these relations back to Eq. (11) after some algebra we arrive at the following non-linear first-order ODE

$$\frac{df}{d\eta^2}\left(\beta^2\eta^2 - f^{2\omega+1}\right) = \frac{\beta f}{2}[f^{\omega+1} - (2\beta+1)]. \tag{16}$$

Put $y=\eta^2$ and $x=f$. With this notation eq. (16) becomes linear for $y(x)$ (this is the second lucky moment of investigation):

$$\frac{dy}{dx} = \frac{y(x) - 4(\omega+1)^2 x^{2\omega+1}}{x[(\omega+1)x^{\omega+1} - \omega - 2]}. \tag{17}$$

Plainly, $f \quad x$ is a solution to eq. (16). If $y(x)$ the solution of eq. (17) is strictly monotonic then so is the inverse function $f=x$ and no discontinuity. However if $y(x)$ is not monotonic on some interval (x_1,

x_2) and has a turning point at $x_0 \varepsilon(x_1, x_2)$ then the inverse ($f=x$) has sense on $[0,y(x)]$ only. One sets $f=0$ for $y> y(x_0)$ and the discontinuity a $y(x_0)$ is apparent. The analytical investigation of the linear equation (17) is, in general easier than of eq (16). In some cases (for some ω s) one can have more explicit or almost explicit solutions.

There are two examples:

The first case is for $\omega=0,(\alpha=1,\beta=1/2,\delta=3/2,\varepsilon=1)$.

This example was studied by [20] in some details. The corresponding ODE (17) reads $y'=(y-4x)/x(x-2)$ which has a solution $y=8+[(x-2)/x]^{1/2}[c_1-8ln(\sqrt{x}+\sqrt{x-2})]$ where c_1 is a constant.

It is clear that must be $x \geq 2$ and $y(x)$ is monotonic for $x>2$ until x_0 where $y=0$. This means that $x(y)$ exists and monotonic on some interval $[0,y_0]$, $x(y_0)=2$; for $y \geq y_0$ we have $x(y)=0$ so the discontinuity. For a better understanding Figure 1a presents the graph of solution of Eq. (17) through the point (3,0.5). The inverse of this function for $x>2$ is shown on Figure 1b (the nonzero part). The solid line is a solution through the $f(0)=10.8$ point. Figure 2 presents the theoretical shock-wave propagation of the temperature distribution $T(r, t)$ for $\omega=0$.

The second case is for $\omega=-1/2,(\alpha=2,\beta=1,\delta=2,\varepsilon=1/2)$.

Now Eq. (17) takes the form of $\frac{dy}{dx}=2(y-1)/[x(\sqrt{x}-3)]$. It can be checked that $y=c_2x^{-2/3}(x^{1/2}-3)^{4/3}$ is a solution for any $c_2>0$. Take $c_{2=0}$. The function $y(x)$ is monotonic on (0,9), $y(9)=0$. Returning to original variables we have $f=9/[(\eta^2)^{3/4}+1]^2$ (which is plainly less than 9!) According to eq. (14) temperature and heat flux distributions are

$$T=\frac{9t}{(r^{3/2}+t^{3/2})^2},\quad 2cmq=\frac{9r}{(r^{3/2}+t^{3/2})^2}. \tag{18}$$

Figure 3 presents the time and the radial coordinate dependence of the temperature and the heat flux. These solutions are not discontinuous. Analytical and numerical calculus suggest that $\omega=-1/2$ is a critical exponent: for $-1<\omega \leq -1/2$ the solutions are continuous, for the shocks always appear $\omega>-1/2$.

Summary

We presented a hyperbolic model for heat conduction in solids where the relaxation time and heat conduction coefficients are power law functions of time. There are basically two different regimes available for different power laws. For $1<\omega \leq -1/2$ the solutions are continuous for all positive time and radial coordinate, for $\omega>-1/2$ the solutions are only continuous on a finite and closed $[0: \eta_{0]}$ interval and have a finite jump at the the endpoint η_0. As physical interpretation numerous

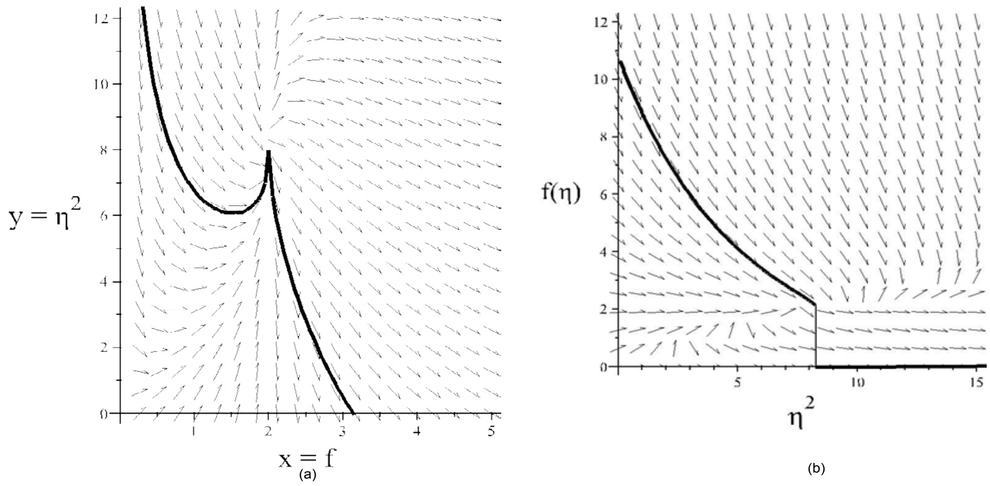

Figure 1: The direction field of (a) Eq. (17) for !=0 and (b) Eq. (16) for!=0 The solid line presents numerical solutions for a) y (3)=0:5 and for b) f (0)=10:8.

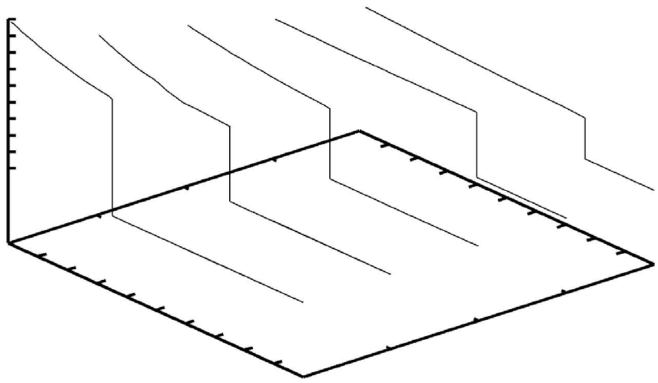

Figure 2: The shock-wave propagation of the temperature distribution of T (r; t) for !=0

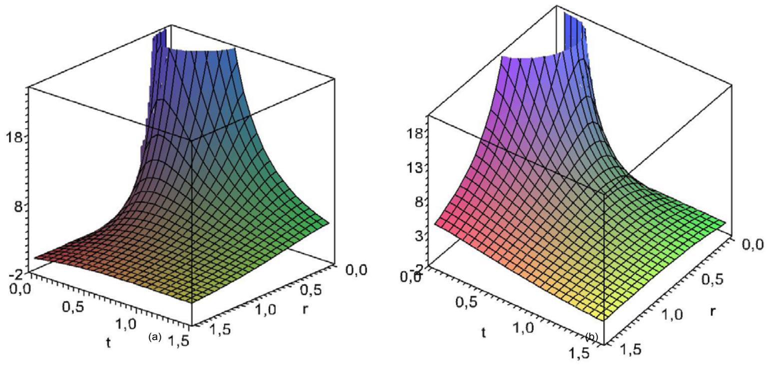

Figure 3: The continuous solutions: (a) the temperature distribution of Eq. (18) and (b) the heat flux distribution of Eq. (18).

materials and solid state systems were mentioned with temperature dependent heat conduction coefficients.

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Citation: Barna IF, Kersner R (2016) Heat Conduction: Hyperbolic Self-similar Shock-waves in Solid Medium. J Generalized Lie Theory Appl S2: 010. doi:10.4172/1736-4337.S2-010

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, Unversity of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications
ISSN: 1736-4337

Karpushkin, J Generalized Lie Theory Appl 2016, S2
http://dx.doi.org/10.4172/1736-4337.S2-011

Research Article Open Access

The ABC's of the Mathematical Infinitology. Principles of the Modern Theory and Practice of Scientific-and-Mathematical Infinitology

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Abstract

The modern Science has now a lot of its branches and meanders, where are working the numerous specialists and outstanding scientists everywhere in the whole world. The theme of this article is devoted to mathematics in general and to such a new subsidiary science as the Cartesian infinitology (± ∞: x y and x y z) in a whole.

The young and adult modern people of our time, among them, in first turn, are such ones as the usual citizens, students or schoolchildren, have a very poor imagination about those achievements and successes that made by our scientists in the different parts and divisions of many fundamental sciences, especially in mathematics. This article is a short description of the numerous ideas of a new science that is named by its inventor as the mathematical infinitology.

The infinity as the scientific category is a very complicated conception and the difficult theme for professional discussing of its properties and features even by the academicians and the Nobelists as well. In spite of all problems, the author have found his own road to this Science and worked out independently, even not being a mathematician at all, the universal, from his point of view, and unusual theories and scientific methods, which helped him to find and name It as the mathematical infinitology, that may be now studied in rectangular system of Cartesian or other coordinates, in orthogonal ones, for example, as easy and practically as we study the organic chemistry or Chinese language at the middle school or in the University.

The mathematical infinitology, as a separate or independent science, has been never existed in the mathematics from the ancient times up to the 90-th years of the XX-th century. All outstanding mathematicians of the past times were able only approximately to image to themselves and explain to their colleagues and pupils in addition, what is an infinity indeed: the scientific abstraction or the natural mathematical science that can be not only tested by one's tooth or touched by hands, but study and investigate it in schools or the Institutions of higher learning too.

In summer 1993, such a specific mathematical object as the "cloth of Ulam", was occasionally re-invented by the article author without no one imagination, what it is indeed. Very long time working hours spent by the inventor with this mathematical toy or the simplest logical entertainment helped him to penetrate into the mysteries of this usual intellectual mathematical object and see in it the fantastic perspectives and possibilities as for science as for himself in further studying and it investigating. In a result of the own purposefulness and interests to the re-invented mathematical idea of the famous American mathematician S.M.Ulam, the new science was born in the World, and after long time experiments, it was named as the mathematical or Cartesian infinitology (±∞ : x y and x y z).

Keywords: Cartesian infinitology; Mathematical plus-minus infinity (±∞:xy and xyz); Cartesian coordinates; Natural prime and twins numbers; Theory of blank spaces; Sieve of Erathosfen; Ulam's spieral

Introduction

In any, praiseworthy hobby, business or the craft, being appeared at the human persons for a long time process of evolution, and thanks to the mental and creative abilities growth, sometimes among the advanced people were developed such high spheres of human knowledge or personal skills or intellectual abilities, that a lot of centuries and even the millenniums came or passed away, before some difficult scientific idea or the secrets of the craft could be at last found their final decisions or they were transformed by the human individuals into such form of the representation or embodiment, available for their natural perception by people, specialists or scientists, that a team of higher skilled experts could only recognize this or that decision as a perfect standard [1-8]. And, it isn't necessary to go far very much for the examples! The most ancient and the unresolved task is a secret of natural prime numbers, the cornerstone of the scientific theory of their knowledge and studying was put by Eratosphen Kirensky, the Ancient Greece mathematician, being lived in the III century B.C. The knowledge by the human persons of the Great truths of the World was always, from the time of immemorial destiny, the elite of possessing advanced thinkers being had a rich life experience. Such people-the unique just always were able and solved the various and most important tasks of their time, advancing thereby not only the era itself and its potential opportunities, but at the same time they were putting by own affairs and talents the progress and forward advance of Mankind on the evolution steps, un-looking on all difficulties and adversities of the daily occurrence, with their terrible wars, epidemics, personal problems and the natural cataclysms [9-15].

And here is already 21 century! It is now improbably interesting to look backward to compare the life of people, which were living at the very beginning of our era, with today's life of people that are living

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Received February 02, 2016; **Accepted** May 27, 2016; **Published** June 03, 2016

Citation: Karpushkin EV (2016) The ABC's of the Mathematical Infinitology. Principles of the Modern Theory and Practice of Scientific-and-Mathematical Infinitology. J Generalized Lie Theory Appl S2: 011. doi:10.4172/1736-4337.S2-011

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now in 2013 A.D. The huge abyss between these two eras is more than evident. Everything was changed considerably and up to beyond recognition! And though the different natural and technogenous misfortunes still annoy to people and their countries, the states and even the whole continents, but what, after all abundance, a huge variety of all forms, and views and types and everything in our civilization! The flights in space and the working Hadron collider became already our daily occurrence [15-21]. And there is already a future man's struggle against the asteroid danger. And the Cheliabinsk fire-ball has showed to the whole world how terrible and dangerous can it be to all living beings on the Earth. It is the most convenient time to think about the security of the Mankind, and its planet too, from the space stone travelers already today. And at soon the possible flights of people to Mars, Venus and other planets of Solar system will be begun. And the wide development of new opportunities of the Arctic and Antarctica areas with their infinite store rooms of minerals and sea bio-sources, in the nearest future! And the problem of shortage of food and drinking water consumption!!! And the catastrophic climate surprises which provoke high-speed thawing of the ice armor of the Earth! The life on the Earth became more unpredictable and dangerous. And in this very quickly changing world, it is difficult to the human person correctly and in due time to react to all misfortunes that are collapsing upon his head from the side of the natural disasters.

Being live rapidly and in the atmosphere of continuous changes, the modern human person, nevertheless, doesn't low his hands down and continues to create the material and intellectual treasures elsewhere on the Earth, and even in the outer space, making better, step by step, not only the created by him achievements but this very complicated World too, on the base of his own imperfections. The people constantly live in continuous creative search, solving the mass of tasks, for what they are sometimes encouraged morally or financially. For the sake of such bright perspectives of the personal wellbeing, the best minds start to look for the solution of the most difficult scientific tasks and other problems. And the valuable awards sometimes find the heroes! This work is a formal confirmation of the man's elementary inquisitiveness and how it helped him to make an interesting scientific invention in sphere of elementary mathematics [22-26].

The Ulam's Cloth or Spiral

Even some a few people among the today's schoolchildren and students know and can convincingly, even on fingers, explain what it is the "Eratosphen's sieve" and/or the "Ulam's spiral", and at least to tell elementarily about these objects, and what it is spoken about in principle. And not all mathematician will be also able to explain objectively and clearly to the ordinary fans of this science, what it is a "bestia" named as the spiral of Ulam, and what are the concrete advantages from it to the science itself, to the ordinary fellow citizens and, especially, to the modern educated people of the world as well [27-30]. If to judge on the single publications only, the mathematical idea of Mr. S. M. Ulam, the famous American mathematician and the Polish man in his original, is not be able to serve as a proof that our authors–educators and the legal distributors of the scientific-and-popular literature on mathematics among the population, have the elementary interest to this, in appearance, the childish mathematical occupation and these persons are not sure very much that they could be objectively and in details to tell for their readers, on the pages of the famous books, about the features of this idea. But what kind of the mathematical interest may have this childish mathematical entertainment at readers in fact?

As it is well known today, Stanislav M. Ulam has invented this "cloth", or rather, a spiral, in 1963, being presented once upon a time at a very boring meeting of his colleges-scientists. To kill time and not to fall asleep with boredom, our hero began to draw on the page of his note-book in cell a symbolic chessboard for solution of etudes, but, occasionally, he has changed his intention and, instead of the chess figures drawing, he begun to fill in the center of this, a poor similarity of the chessboard, with the natural prime numbers in view of the points situated in square cells of the spiral-typed line, turning anticlockwise, that replaced such prime numbers as two, three, etc. As for me, I have made the same even not being introduced with this idea at all and its author in general [30-37]. Both Ulam and me have replaced the prime numbers with the points for simplification of the whole work. And at soon, the idea of the American mathematician, which was named as "the Ulam's cloth" by the scientists, was born and, by the time, it has possessed the right to live. Specialists of Los-Alamos laboratory, headed by Stanislav Martin Ulam, the author of this idea, did a huge work on detection the regularities of prime numbers distribution within this helicoid system, but the idea, as it is known, couldn't demonstrate itself in its entire beauty since it was needed a perfect modification a little. But just on this trifle, the time was absent at S.Ulam and his colleges. So it's a pity! Because Stanislav Martin Ulam and his friends in this laboratory have been on the threshold of the Great discovery in mathematics, and, as it is supposed by me, in sphere of the elementary number theory [38-44].

The spiral of Ulam

82 **81** 80 **79** 78 77 76 75 74 **73**

83 50 **49** 48 **47** 46 45 44 **43** 72

84 51 26 **25** 24 **23** 22 21 42 **71**

85 52 27 10 **09** 08 07 20 **41** 70

86 **53** 28 **11** **02** **01** 06 **19** 40 69

87 54 **29** 12 03 **04** 05 18 39 68

88 55 30 **13** 14 15 **16** **17** 38 **67**

89 56 **31** 32 33 34 35 **36** **37** 66

90 57 58 **59** 60 **61** 62 63 **64** 65

91 92 93 94 95 96 **97** 98 99 **100**…→ ∞

Even such a small site of mathematical object under the name "Ulam's cloth" allows seeing the fine accurate chains of the natural numbers-points on the Figures 1-3 below and in the [LI]

Classification of the Natural Numbers in the "Spiral of Ulam"

I. 1,2,3,4,5,6,7,8,9,10,11,12,…--- the usual natural numbers consequence;

I.' 1,3,5,7,9,11,13,15,17,19,21 --- the odd natural numbers consequence,…

I." 2,4,6,8,10,12,14,16,18,20,…--- the even natural numbers consequence,

II. 2,3,5,7,11,13,17,19,23,31,…--- the natural prime numbers consequence;

III. 3-5,5-7,11-13,29-31,41-43,…--- the natural twin numbers consequence;

IV. 1,9,25,49,81,121,169,196,… --- the squares of the odd natural numbers consequence;

V. 4,16,36,64,100,144,196,... --- the squares of the even natural numbers consequence;

VI. 1,11,31,41,61,71,101,131,... --- the first kin type of prime numbers consequence;

VII. 3,13,23,43,53,73,83,103,... --- the second kin type of prime numbers consequence;

VIII. 7,17,37,47,67,97,107,127,... --- the third kin type of prime numbers consequence;

IX. 19,29,59,79,109,139,149,... --- the forth kin type of prime numbers consequence;

X. 1,3,11,13,23,31,41,43,53,... --- the fifth kin type of prime numbers consequence;

XI. 1,7,11,17,31,37,41,47,61,... --- the sixth kin type of prime numbers consequence;

XII. 1,11,19,29,31,41,59,61,71,...--- the seventh kin type of prime numbers consequence;

XIII. 3,7,13,17,23,37,43,47,53,... --- the eighth kin type of prime numbers consequence;

XIV. 3,13,19,23,29,43,53,59,73,...--- the ninth kin type of prime numbers consequence;

XV. 7,17,19,29,37,47,59,67,79,...--- the tenth kin type of the prime numbers consequence;

XVI. 1,3,7,11,13,17,23,31,37,41,...--- the eleventh kin type of prime numbers consequence;

XVII. 1,3,11,13,19,23,29,31,41,...--- the twelfth kin type of prime numbers consequence;

XVIII.1,7,11,17,19,29,31,37,41,... --- the thirteenth kin type of prime numbers consequence;

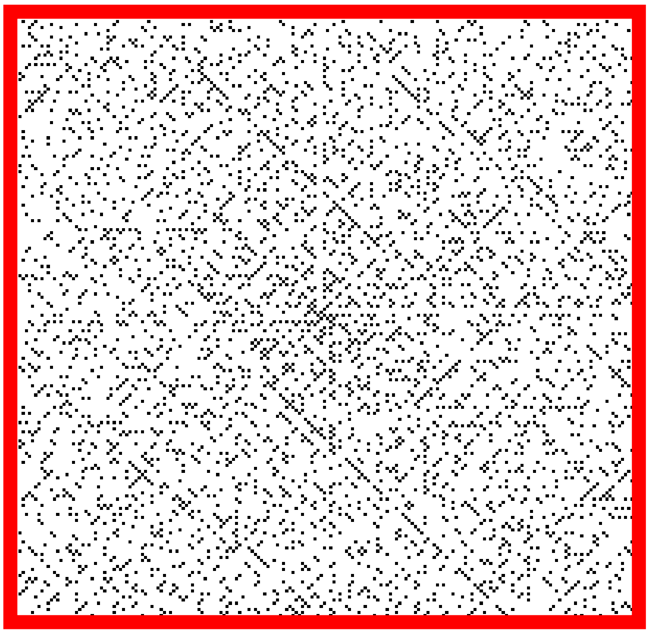

Figure 1: The mathematical rectangular spiral or "the table-cloth of Ulam" (fragment). (the black points are the symbol prime numbers on the white field)

XIX. 3,7,13,17,19,23,29,37,43,... --- the fourteenth kin type of prime numbers consequence;

XX. 11-13,41-43,71-73,101-103,...--- the 1-st kin of the twin prime numbers consequence;

XXI. 17-19,107-109,137-139,... --- the 2-nd kin of the twin prime numbers consequence;

XXII. 29-31,59-61,149-151,... --- the 3-d kin of the twin prime numbers consequence;

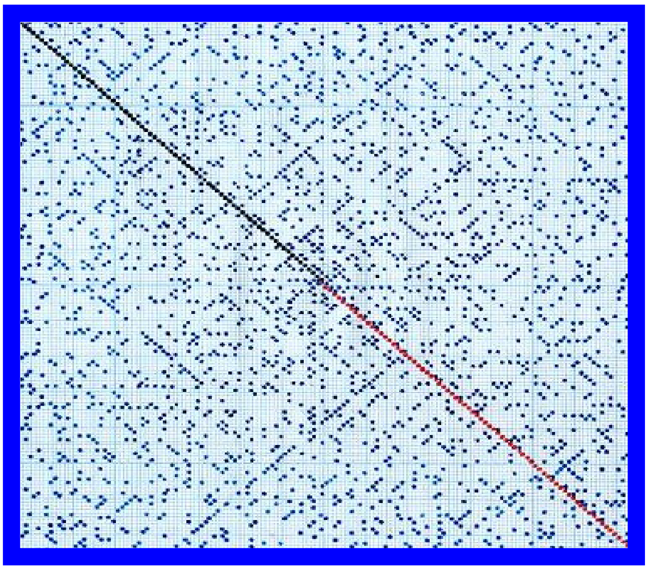

Figure 2: The mathematical rectangular spiral or "the table-cloth of Ulam" (fragment), (the white points are the symbol prime numbers on the black field).

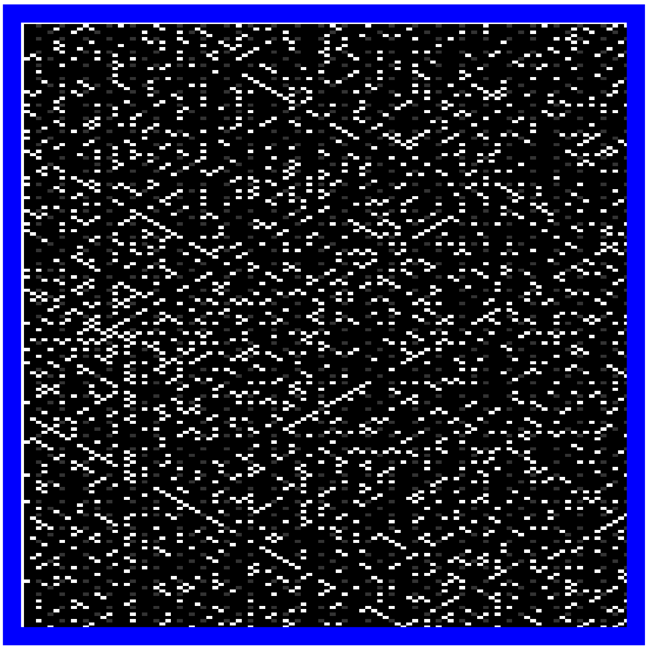

Figure 3: The generalized mathematical rectangular spiral or "the table-cloth of Ulam" (the dark-blue points are the symbol prime numbers; the green points are the symbol odd quadratic numbers; the red points are the symbol even numbers on the white field. The fig. was made by the Author of the article after careful number coordinates calculation).

XXIII. 11-13,17-19,41-43,71-73,…--- the 4-th kin of the twin prime numbers consequence;

XXIV. 11-13,29-31,41-43,59-61,…--- the 5-th kin of the twin prime numbers consequence;

XXV. 17-19,29-31,59-61,107-109, --- the 6-th kin of the twin prime numbers consequence (Figure 1-3).

The "Ulam's cloth" accurate chains of the natural numbers and their analogs in view of sets of the same dots on the different color fields (white, light-blue and black)) are demonstrating the verible variants of regularity of the natural number distribution in the spiral of Ulam. But if to look at this peculiar roll from numbers indifferently, of course, it is nothing interesting will be found in this spiral. For those fifty years, which have passed from that day, when Stanislav Ulam has invented this "toy", which wasn't attracted only by it to check up one's intellect and satisfy one's vanity playing with this, in appearance, the usual ordinary numerical spiral! But nobody was able to see or understand it's most important and basic features [45-51]. May be this "cat in the bag" was "sitting" there up to the end of the times on a scientific shelf or in corner of the old store-room or a hose or rectangular "boa" rounded tightly and forgotten by everybody forever, if once upon a time, exactly twenty years ago, the author of these lines also decided but occasionally to solve one simple arithmetic problem. In the course of its decision, when all known methods were tried without results, suddenly the entertainment of my student's years came to my mind---a mathematical rectangular spiral, which sometimes should be drawn by me at very boring lectures. In my student years during the boring lectures, I created the spiral of natural numbers and marked the natural prime numbers situated in cells of it, on the page of my student's note-book exactly as it was made by Stanislav Ulam, (that I know much later, having looked through the mountains of mathematical literature). I have been already ready to end my empty occupations with this spiral. When I wanted to find the possible decision of my arithmetic task, when, at the last moment, I have noticed one strangeness, which strongly intrigued and surprised me: I noticed, that all squares of odd natural numbers at this spiral ideally correctly were situated on the diagonal leaving the center of this spiral and gone to the left corner, but the squares of even natural numbers---to the opposite side of the spiral (Figure 2) [52-59].

And then a great willing has come to my mind-to fulfill the graphical generalization of this elementary spiral. But to do so, one ought to me to make a huge volume of calculations and graphical works. And for the aim to receive a fine and interesting picture-beautiful and demonstrated one, it has been decided to mark the suitable natural numbers with the dots of the corresponded color. In a result, the natural prime and twin numbers have been coded with the dots of blue-dark color, the squares of the odd natural numbers have become the green and the squares of the even natural numbers and the null too-the red ones. Such simple color coding or marking of the natural numbers have made the powerful and strong basement for a new scientific idea and the future new mathematical science. And later, after deep studying of it, this idea has been named as the "Generalized spiral of Ulam". It is graphical interpretation is shown on the Figures 2 and 4.

82 **81** 80 **79** 78 77 76 75 74 **73**

83 50 **49** 48 **47** 46 45 44 **43** 72

84 51 26 **25** 24 **23** 22 21 42 **71**

85 52 27 10 **09** 08 0**7** 20 **41** 70

86 **53** 28 **11** **02** **01** 06 **19** 40 69

87 54 **29** 12 0**3** **04** 0**5** 18 39 68

88 55 30 **13** 14 15 **16** **17** 38 **67**

89 56 **31** 32 33 34 35 **36** **37** 66

90 57 58 **59** 60 **61** 62 63 **64** 65

91 92 93 94 95 96 **97** 98 99 **100**…→ ∞

Figure 4: Generalized "the table - cloth of Ulam".

"Generalized Spiral of Ulam"

Analogs and derivations of the generalized spiral of Ulam

At once and immediately, when was determined the main information about such a strange and even the mysterious scientific object as the spiral of Ulam, there were begun the longest searching of more detailed descriptions of such spiral in the suited editions, publications, and manuals on mathematics. But having reconsidered the hills of books and handbooks on the elementary and higher mathematics, I was not able to find the information about this neither the spiral nor the generalized analog of it. Having supposed that this idea has not even the elementary interest and attention at the mathematicians, I begun to study this "toy" independently, being made my own varieties of this spiral for differentiation of my own entertainment only. In a result of my interactivity, the most improbable compositions have begun to appear from the natural numbers, which, after replacement the natural numbers on the color dots, I have received their own names like these ones: triangular, trapeziform, zigzag, and so on. There are some types and kinds of such number compositions below, that have been created on the base of my big interest and my own version of the generalized spiral of Ulam too (Figures 5-10).

If to look attentively and carefully at the natural number compositions, we then will not be able to un-notice a new and very interesting feature --- the square powers of the odd and even natural numbers, as usual, have created again their special configurations and

21

22 7 20

23 8 **1** 6 19

24 **9** 2 3 **4** 5 18

25 1 11 12 13 14 15 **16** 17

26 27 28 29 30 31 32 33 34 35 **36**

Figure 5: Triangular spiral. {An}={n}.

50 **49**

51 26 **25** 48

52 27 10 **9** 24 47

53 28 11 2 **1** 8 23 46

54 29 12 3 **4** 5 6 7 22 45

55 30 13 14 15 **16** 17 18 19 20 21 44

56 31 32 33 34 35 **36** 37 38 39 40 41 42 43

Figure 6: Trapeziform spiral. {An}={n}.

Citation: Karpushkin EV (2016) The ABC's of the Mathematical Infinitology. Principles of the Modern Theory and Practice of Scientific-and-Mathematical Infinitology. J Generalized Lie Theory Appl S2: 011. doi:10.4172/1736-4337.S2-011

such a manner, that the noticed at the Generalized spiral of Ulam unordinal peculiarity to form their individual sets and subsets in view of the consequent chains of red and green dots, is nowhere broken in its new verities (Figures 3-9). Such a peculiarity is more persuasive than any words can say that perhaps a new and nobody known property of usual natural numbers is found in mathematics. The further investigations of this property, discovered at the natural numbers, allowed recognizing it as the universal low at them and at their algebraic-and-complex equivalents as well, and it has been officially registered in the State notary office, in the Murmansk Regional town center, situated on Kola Peninsula, in Russia.

131
109 129
89 107 127
71 87 105 125
55 69 85 103 123
41 53 67 83 101 **121**
29 39 51 65 **81** 99 119
19 27 37 **49** 63 79 97 117
11 17 **25** 35 47 61 77 95 115
05 **09** 15 23 33 45 59 75 93 113
01 03 07 13 21 31 43 57 73 91 111

Figure 7: Zigzag spiral. {An}={2n – 1}

99 **100**
80 **81** 98 101
63 **64** 79 82 97 103
48 **49** 62 65 78 83 96 104
35 **36** 47 50 61 66 77 84 95 105
24 **25** 34 37 46 51 60 67 76 85 94 106
15 **16** 23 26 33 38 45 52 59 68 75 86 93 107
8 **9** 14 17 22 27 32 39 44 53 58 69 74 87 92 108
3 **4** 7 10 13 18 21 28 31 40 43 54 57 70 73 88 91 109
0 **1** 2 5 6 11 12 19 20 29 30 41 42 55 56 71 72 89 90 110

Figure 8: Serpentine spiral. {An}={n}.

223 221 219 217 215 213 211 209 207 205 203 201 199 197 195
225 143 145 147 149 151 153 155 157 159 161 163 165 167 193
227 141 119 117 115 113 111 109 107 105 103 101 99 **169** 191
229 139 **121** 63 65 67 69 71 73 75 77 79 97 171 189
231 137 123 61 47 45 43 41 39 37 35 **81** 95 173 187
233 135 125 59 **49** 15 17 19 21 23 33 83 93 175 185
235 133 127 57 51 13 7 5 3 **25** 31 85 91 177 183
237 131 129 55 53 11 **9** 0 **1** 27 29 87 89 179 181

Figure 9: Funnel-shaped (vortex) spiral. {An}={2n–1}.

Triangular structure

When, as it was seemed, the all possible variants and varieties of the Generalized spiral of Ulam were invented and compiled, it is naturally the idea has appeared to create a new natural number configuration in view e.g. of pyramid or isosceles rectangular triangle, standing on one of its sides (Figure 11). In a new variant one more variety of the Generalized spiral of Ulam, it suddenly has been discovered that the spiral of Ulam, written in such a manner, is principally differ from its previous variants on the external view and other parameters (i.e. red and green dots had other configurations at the schematic diagram). In this triangular structure were seen clearly the counters of the famous and well-known to everyone in mathematics the second order curve - the parabola itself (Figures 11 and 12).

Graph-and-Analytical Method

Standard variant

Let us write in common view the consequence of derivation

90
91
24 32 48 54 62 74 84 92
25 33 **49** 55 63 75 85 93
8 14 20 26 34 38 44 50 56 **64** 68 76 80 86 94 98
0 **9** 15 21 27 35 39 45 51 57 65 69 77 **81** 87 95 99
1 **4** 6 10 12 **16** 18 22 28 30 **36** 40 42 46 52 58 60 66 70 72 78 82 88 96 **100**
2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 71 73 79 83 89 97 101

Figure 10: "The New-York silhouette". (An}={n}.

01
02 03
04 05 06
07 08 **09** 10
11 12 13 14 15
16 17 18 19 20 21
22 23 24 **25** 26 27 28
29 30 31 32 33 34 35 **36**
37 38 39 40 41 42 43 44 45
46 47 48 **49** 50 51 52 53 54 55
56 57 58 59 60 61 62 63 **64** 65 66
67 68 69 70 71 72 73 74 75 76 77 78
79 80 **81** 82 83 84 85 86 87 88 89 90 91
92 93 94 95 96 97 98 99 **100** 101 102 103 104 105
106 107 108 109 110 111 112 113 114 115 116 117 118 119 120
121 122 123 124 125 126 127 128 129 130 131 132 133 134 135 136
137 138 139 140 141 142 143 **144** 145 146 147 148 149 150 151 152 153

Figure 11: Triangular stepped structureю. {An}={n}.

Citation: Karpushkin EV (2016) The ABC's of the Mathematical Infinitology. Principles of the Modern Theory and Practice of Scientific-and-Mathematical Infinitology. J Generalized Lie Theory Appl S2: 011. doi:10.4172/1736-4337.S2-011

1
x x
4 x x
x x 9 x
x x x x x
16 x x x x x
x x x 25 x x x
x x x x x x x 36
x x x x x x x x x
x x x 49 x x x x x x
x x x x x x x x 64 x x
x x x x x x x x x x x x
x x 81 x x x x x x x x x x
x x x x x x x x 100 x x x x x
x x x x x x x x x x x x x x x
121 x x x x x x x x x x x x x x x
x x x x x x x 144 x x x x x x x x x

Figure 12: Triangular stepped structure. {An}={n}.

of the second order line equation or the algebraic curve placed in Cartesian coordinates and going through the five coordinate points. When used with this method, one can calculate all types of polynomials and algebraic equations of all quadratic parabolas, the thin contours of which are formed by the sets of red, red-green and green dots on the plot [L1] of the natural numbers [{An}={n²}] - type consequence in the rectangular system of Cartesian coordinates [60-65].

The equation of the algebraic curve of the second order, that going through the five points: M_1 (x_1, y_1); M_2 (x_2, y_2); M_3 (x_3, y_3); M_4 (x_4, y_4) и M_5 (x_5, y_5), one can calculate it with the following method, that well-known in mathematics as the Method of determinants:

Let us write four determinants and their algebraic equalities:

$$M_1M_2\text{: } A(x, y)=\begin{vmatrix} x & y & 1 \\ x & y_1 & 1 \\ x & y_2 & 1 \end{vmatrix}=0$$

$$A(x, y)=xy_1+x_2y+x_1y_2-x_2y_1-xy_2-x_1y$$

$$M_2M_3\text{: } B(x, y)=\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$$

$$B(x, y)=xy_2+x_3y+x_2y_3-x_3y_2-xy_3-x_2y$$

$$M_3M_4\text{: } C(x, y)=\begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}=0$$

$$C(x, y)=xy_3+x_4y+x_3y_4-x_4y_3-xy_4-x_3y$$

$$M_4M_1\text{: } D(x, y)=\begin{vmatrix} x & y & 1 \\ x_4 & y_4 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}=0$$

$$D(x, y)=xy_4+x_1y+x_4y_1-x_1y_4-xy_1-x_4y$$

Let us write the equation:

$$P.A(x, y)\cdot C(x, y)+Q.B(x, y).D(x, y)=0, \qquad (1)$$

where P and Q – any real numbers that are not equal to zero simultaneously. Let us find such

a relation of P and Q that M_5 has become to belong to the line (1).

$$P: Q=[(-B)(x_5, y_5)\cdot D(x_5, y_5)] : [A(x_5, y_5)\cdot C(x_5, y_5)] \qquad (2)$$

Let us find the meanings of P and Q and then insert these meanings in (1) and then we will try to decide this equation. After collecting like terms, we will have the algebraic equation of the second order line that going through the five known points [66,67].

In view of the practical example, let us calculate the equation of the second order line, the main points of which are situated in the negative area of the coordinate axis (–XoX+), having determined the meaning of coordinates of this curve with the help of the plot [L1], where we will find easily the first five green points of the furthest parabola, the symmetrical axis of which is parallel to the (–XoX+) coordinate line and combines with it.

M_1 (– 130½; 4½); M_2 (– 126½; 3½); M_3 (– 123½; 2½); M_4 (– 121½; 1½); M_5 (– 120½; ½)

Let us find the mediate equations and suited coefficients for derivation of the desired algebraic equation or the second order line, going through the five given points.

$$M_1M_2\text{: } A(x, y)=\begin{vmatrix} x & y & 1 \\ (-261/2) & 9/2 & 1 \\ (-253/2) & 7/2 & 1 \end{vmatrix}=0$$

$$A(x, y)=(9/2)x-(253/2)y-1827/4+2277/4-(7/2)x+(261/2)y$$

$$\rightarrow 2x+8\,y+225=0$$

$$M_2M_3\text{: } B(x, y)=\begin{vmatrix} x & y & 1 \\ (-253/2) & 7/2 & 1 \\ (-247/2) & 5/2 & 1 \end{vmatrix}=0$$

$$B(x, y)=(7/2)\,x-(247/2)\,y-1265/4+1729/4-(5/2)\,x+(253/2)\,y$$

$$\rightarrow x+3y+116=0$$

$$M_3M_4\text{: } C(x, y)=\begin{vmatrix} x & y & 1 \\ (-247/2) & 5/2 & 1 \\ (-243/2) & 3/2 & 1 \end{vmatrix}=0$$

$$C(x, y)=(5/2)\,x-(243/2)y-741/4+1215/4-(3/2)x+(247/2)y$$

$$\rightarrow 2x+4y+237=0$$

$$M_4M_1\text{: } D(x, y)=\begin{vmatrix} x & y & 1 \\ (-243/2) & 3/2 & 1 \\ (-261/2) & 9/2 & 1 \end{vmatrix}=0$$

D(x, y)=(3/2) x-(261/2)y – 2187/4+783/4 – (9/2)x+(243/2)y

→ 6x+18y+702=0

6.1.6. Let us write the desired equation:

$$P\,(2x+8y+225)(2x+4y+237)+Q\,(x+3y+116)(6x+18y+702)=0 \qquad (3)$$

Let us find such a relation of P:Q that the $M_5(-120½;½)$ point became to belong to this line:

x+3y+116=0 (–241/2)+3/2+232/2=(–3)

6x+18y+702=0 (–1446/2)+18/2+1404/2=(–12)

2x+8y+225=0 (– 241)+4+225=(–12)

2x+4y+237=0 (– 241)+2+237=(–2)

(P/Q)=[–(–3)(–12)]/[(–12)(–2)] (P/Q)=(–3)/2

P=(–3) Q=2

Let us open the brackets in the equality (3) and then collect like terms with taking into account the meaning of the P and Q coefficients:

(–3)(2x+8y+225)(2x+4y+237)+2(x+3y+116) (6x+18y+702)=0

$-12x^2 - 72xy - 2772x - 96y^2 - 8388 - 159975=0$

$+\{12x^2+72xy+2796x+108y^2+8388+162864=0$

$$12\,y^2+24\,x+2899=0 \qquad (4)$$

Let us determine the coordinates of M_0 top of the parabola (4). Let us y_0=0.

12 · 0+24x+2889=0 24x=–2889; x_0=– 120 ⅜ y_0=0

By turning the X and Y axes on (± 90º) and (±180°) around the null-point of the Cartesian coordinates, we then will have four main quadratic equations:

$12y^2 \pm 24\,x+2889=0$ $y=\pm(½\,x^2+120\,⅜)$

Offered here calculation presents the famous method of determining the polynomials of those classical or created by the mathematical Nature of idea itself of the algebraic equations, the assemblage of coordinate points of which forms the interminable two-color dotted plot [L1] of the $\{A_n\}=\{n^2\}$-type natural numbers consequence in the rectangular system of Cartesian coordinates in the given scale and intervals alongside the X and Y axes and far from them on the unlimited fields of the rectangular system of Cartesian coordinates [68].

Universal Classifier of the Natural Numbers and Its Varieties

For successful continuation of the natural numbers studying and investigation them in the limits of this idea, the necessity has suddenly appeared how to find or invent independently the universal and simplest method of the natural numbers classification. After very long and difficult seeks, it was invented at last such a numerical clepsydra or mathematical sieve that was able to characterize any natural number in view of its simplest parameters like these ones: evenness, oddness, simplicity, divisibility, etc. The universal mathematical natural numbers detector has been invented at last in the mathematical science.

In fact, the Universal classifier itself is a usual table in view of the right isosceles triangle that is widened to its horizontal side, and where the natural numbers are consequently roomed in cells from the point of their division on all possible whole devisors. It is the Universal classifier of the natural numbers that now allows to decide all simplest tasks on the any natural numbers parameterization. The Universal classifier itself and its varieties are placed below. When analyzing the Classifier structure and its principles of working, one can easily to see and understand the real Classifier's advantages in comparison with the analogical mathematical tables and schematic diagrams. It is a natural mini-mathematical Encyclopedia under ones hand.

In all times, there were people that tried to classify all and everything in the World. In a result, all people's achievements have begun to undergo to the common and the all-world classification. Each branch or direction of the human activity were analyzing by their pioneers or outstanding scientists. The Mankind, thanks to such clever persons, has possessed a dozen of sciences and their numerous meanders. All things, even the tiniest elements of them, have now their shelf, place or cell in the Great archives, created by the people [69,70].

As for our ideas and methods of classification the rules and lows for creating the correct different dotted color illustrations, pictures and plots (graphs) in Cartesian coordinates, this science or mathematical infinitology requires the strong and ideal classification of all aspects and ideas in this huge scientific sphere. Having constructed the powerful base for such a complicated science like the mathematical infinitology, we must be sure that our system of rules, axioms, classifications and scientific imaginations, will be strong and undestroyed forever (Figures 13-16).

After a successful creating of three dotted multi-colored graphs and plots in the rectangular system of Cartesian coordinates, the most unusual and interesting idea has born suddenly as "Eureka !" at Archimedes. It suddenly dawned upon me and the main result of such a premonition, presented here like the Classification table, was the idea of creating the dotted scientific illustration, the mathematical interpretation or close similarity to it would be a formula {An}={n}, where the "n" is any natural number, marked in a view of the dot having the only possible color for this figure. But because of the absence of the axioms and the already written rules on the natural numbers color coding, this idea has taken me unawares, and I was needed, by all means, to find, invent or work out such method of natural number color coding at once, immediately and independently.

Later, in a result of the purposefulness and own interest, this difficult task was decided in the shortest time and much enough successfully. To be more specific, any natural number in the endless consequence of them, one can code (mark) now with the only color, and as for any figure color marking, it will be needed only seven "paints" of the rainbow spectrum for these purposes. The rules of color natural number coding is presented here in the Classification table below, taking, of course, in our mind, that each figure on the picture or a plot is represented there in a view of the suited color dot. Let us introduce with the elementary color coding rules of the natural numbers and their complex-and-algebraic equivalents as well [71-75].

Classification Table

1. Any odd natural number, arisen in "() ^2" or any other "() ^2n" power, is coded in view of the green dot(s), e.g.: 1^2=1, 3^2=9, 5^2=25, etc. The same but the negative odd numbers (-1, -9, -25, etc.) must be marked in such a manner, i.e. in view of the green dot(s) on the plot or graph, created in the Cartesian coordinates.
2. Any even natural number, arisen in "() ^2" or any other "() ^2n" power, is coded in view of the red dot(s), e.g.: 2^2=4**,** 4^2=16**,** 6^2=36, etc. The same but the negative even numbers (-4, -16,

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0																			
1	1																		
2	1	2																	
3	1	–	3																
4	1	2	–	4															
5	1	–	–	–	5														
6	1	2	3	–	–	6													
7	1	–	–	–	–	–	7												
8	1	2	–	4	–	–	–	8											
9	1	–	3	–	–	–	–	–	9										
10	1	2	–	–	5	–	–	–	–	10									
11	1	–	–	–	–	–	–	–	–	–	11								
12	1	2	3	4	–	6	–	–	–	–	–	12							
13	1	–	–	–	–	–	–	–	–	–	–	–	13						
14	1	2	–	–	–	–	7	–	–	–	–	–	–	14					
15	1	–	3	–	5	–	–	–	–	–	–	–	–	–	15				
16	1	2	–	4	–	–	–	8	–	–	–	–	–	–	–	16			
17	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	17		
18	1	2	3	–	–	6	–	–	9	–	–	–	–	–	–	–	–	18	
19	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	19

Figure 13: Elementary classifier of the natural numbers.

0																														
2	1	2																												
4	1	2	–	4																										
6	1	2	3	–	–	6																								
8	1	2	–	4	–	–	–	8																						
1	1	2	–	–	5	–	–	–	–	1																				
0										0																				
1	1	2	3	4	–	6	–	–	–	–	–	1																		
2												2																		
1	1	2	–	–	–	–	7	–	–	–	–	–	–	1																
4														4																
1	1	2	–	4	–	–	–	8	–	–	–	–	–	–	–	1														
6																6														
1	1	2	3	–	–	6	–	–	9	–	–	–	–	–	–	–	–	1												
8																		8												
2	1	2	–	4	5	–	–	–	–	1	–	–	–	–	–	–	–	–	–	2										
0										0										0										
2	1	2	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	2								
2											1											2								
2	1	2	3	4	–	6	–	8	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	2						
4												2												4						
2	1	2	–	–	–	–	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	2					
6													3												6					
2	1	2	–	–	–	–	7	–	–	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	2		
8														4														8		
3	1	2	3	–	5	6	–	–	–	1	–	–	–	–	1	–	–	–	–	–	–	–	–	–	–	–	–	–	3
0										0					5														0

Figure 14: Elementary classifier of the even natural numbers.

Citation: Karpushkin EV (2016) The ABC's of the Mathematical Infinitology. Principles of the Modern Theory and Practice of Scientific-and-Mathematical Infinitology. J Generalized Lie Theory Appl S2: 011. doi:10.4172/1736-4337.S2-011

0																		
1	1																	
3	1	3																
5	1	–	5															
7	1	–	–	7														
9	1	3	–	–	9													
11	1	–	–	–	–	11												
13	1	–	–	–	–	–	13											
15	1	3	5	–	–	–	–	15										
17	1	–	–	–	–	–	–	–	17									
19	1	–	–	–	–	–	–	–	–	19								
21	1	3	–	7	–	–	–	–	–	–	21							
23	1	–	–	–	–	–	–	–	–	–	–	23						
25	1	–	5	–	–	–	–	–	–	–	–	–	25					
27	1	3	–	–	9	–	–	–	–	–	–	–	–	27				
29	1	–	–	–	–	–	–	–	–	–	–	–	–	–	29			
31	1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	31		
33	1	3	–	–	–	11	–	–	–	–	–	–	–	–	–	–	33	
35	1	–	5	7	–	–	–	–	–	–	–	–	–	–	–	–	–	35

Figure 15: Elementary classifier of the odd natural numbers.

00																		
01	01																	
03	01		03															
06	01	02	03			06												
09	01		03						09									
12	01	02	03	04		06						12						
15	01		03		05										15			
18	01	02	03			06			09								18	
21	01		03				07											21
24	01	02	03	04		06		08				12						
27	01		03						09									
30	01	02	03		05	06				10					15			
33	01		03								11							
36	01	02	03	04		06			09			12					18	
39	01		03										13					
42	01	02	03			06	07							14				21
45	01		03		05				09						15			
48	01	02	03	04		06		08				12				16		

Figure 16: Elementary classifier of the {An}=3n view natural numbers.

-36, etc.) must be marked in such a manner, i.e. in view of the red dot(s) on the plot or graph, created in the Cartesian coordinates.

3. Any natural prime or twin numbers must be coded in view of the blue dot(s), e.g.: 2=2, 3=3, 5=5, etc. The same but the negative numbers (-2, -3, -5, etc.) must be marked in such a manner, i.e. in view of the blue dot(s) on the plot or graph, created in the Cartesian coordinates.
4. Any odd natural number, arisen in "() ^3" or any other "() ^(2n-1)" power, is coded in view of the dark blue dot(s), e.g.: $3^3=27$, $5^3=125$, $7^3=343$, etc. The same but the negative numbers (-27, -125, -343, etc.) must be marked in such a manner, i.e. in view of the dark blue dot(s) on the plot or graph, created in the Cartesian coordinates, excluding the natural (negative) numbers, which fall under the condition of the item No.1 of this Classification, e.g.: $9^3=[(3^2)]^3$, etc.
5. Any even natural number, arisen in "() ^3" or any other "() ^(2n-1)" power, is coded in view of the violet dot(s), e.g.: $2^3=8$, $6^3=216$, 2^9=512, etc. The same but the negative numbers (-8, -216, -512, etc.)must be marked in such a manner, i.e. in view of the violet dot(s) on the plot or graph , created in the Cartesian coordinates, excluding the natural (negative) numbers, which fall under the condition of the item No. 2 of this Classification, e.g.: $4^3=[(2^2)]^3$, etc.
6. All other odd natural (negative) numbers are coded in view of the yellow dot(s), e.g.:15, 21, 33, 35, 39, 45, 51, 55, etc., when created the plot or graph in Cartesian coordinates.
7. All other even natural (negative) numbers are coded in view of the orange dot(s), e.g.: 6, 10, 12, 14, 18, 20, 22, 24, etc., when created the plot or graph in Cartesian coordinates.

Such a simple method of any natural number color classification in a view of the dot, having the own color among the seven paints of the rainbow spectrum, will allow to create for us not only the most unusual scientific and art "pictures" but even the fantastic dotted illustrations and compositions in the rectangular system of Cartesian coordinates in the vicinity of its "null"-point and at any distance from it. The modern programmable media products such ones of them as MAPLE, MathCAD, MATHEMATICA, MATLAB, WOLFRAM, etc., will help to strength the opportunities for our scientists-mathematicians and specialists in sphere of IBM PC programming up to the endless indeed.

And, probably, some new scientific inventions will be made as in mathematics as in physics, chemistry, astronomy and other famous sciences and their branches. And, may be, at last, the mathematical or Cartesian plus-minus infinity (± ∞ : x y and x y z) will tell to its investigators all secrets of the prime numbers, twin numbers, proof the conjunction of Riemann B and explain a lot of other outstanding scientific and mathematical problems of the past centuries and modern ones additionally.

Combinatorics

Variants of color coding of natural numbers and formed by them consequences

After working out the principles of natural numbers color coding in the limits of this idea, it has appeared the possibility to make and create as manually as electronically the most variable, dependent on their chromaticity and color compositing the dotted illustrations and pictures or scientific dotted - colored graphs of the natural numbers and formed by them consequences in the rectangular system of Cartesian coordinates.

One – color graphs:

1. Green (gr) 2. Red (rd) 3. Blue (bl) 4. (c) Light blue (lb) 5. Violet (vt) 6. Yellow (yl) 7. Orange (rn)

C=7!/[1! (7 – 1) !] C=7

Two-color graphs:

1. 1-2	2. 1-3	3. 1-4	4. 1-5	5. 1-6	6. 1-7	7. 2-3
8. 2-4	9. 2-5	10. 2-6	11. 2-7	12. 3-4	13. 3-5	14. 3-6
15. 3-7	16. 4-5	17. 4-6	18. 4-7	19. 5-6	20. 5-7	21. 6-7

C=7!/[2! (7 – 2) !] C=21

Three-color graphs:

1. 1-2-3	2. 1-2-4	3. 1-2-5	4. 1-2-6	5. 1-2-7	6. 1-3-4	7. 1-3-5
8. 1-3-6	9. 1-3-7	10. 1-4-5	11. 1-4-6	12. 1-4-7	13. 1-5-6	14. 1-5-7
15. 1-6-7	16. 2-3-4	17. 2-3-5	18. 2-3-6	19. 2-3-7	20. 2-4-5	21. 2-4-6
22. 2-4-7	23. 2-5-6	24. 2-5-7	25. 2-6-7	26. 3-4-5	27. 3-4-6	28. 3-4-7
29. 3-5-6	30. 3-5-7	31. 3-6-7	32. 4-5-6	33. 4-5-7	34. 4-6-7	35. 5-6-7

C=7 !/[3! (7 – 3) !] C=35

Four-color graphs:

1. 1-2-3-4	2. 1-2-3-5	3. 1-2-3-6	4. 1-2-3-7
5. 1-2-4-5	6. 1-2-4-6	7. 1-2-4-7	8. 1-2-5-6
9. 1-2-5-7	10. 1-2-6-7	11. 1-3-4-5	12. 1-3-4-6
13. 1-3-4-7	14. 1-3-5-6	15. 1-3-5-7	16. 1-3-6-7
17. 1-4-5-6	18. 1-4-5-7	19. 1-4-6-7	20. 1-5-6-7
21. 2-3-4-5	22. 2-3-4-6	23. 2-3-4-7	24. 2-3-5-6
25. 2-3-5-7	26. 2-3-6-7	27. 2-4-5-6	28. 2-4-5-7
29. 2-4-6-7	30. 2-5-6-7	31. 3-4-5-6	32. 3-4-5-7
33. 3-4-6-7	34. 3-5-6-7	35. 4-5-6-7	36. 0-0-0-0

C=7!/[4! (7 – 4)!] C=35

Five-color graphs:

1. 1-2-3-4-5	2. 1-2-3-4-6	3. 1 -2-3-4-7	4. 1-2-3-5-6
5. 1-2-3-5-7	6. 1-2-3-6-7	7. 1-2-4-5-6	8. 1-2-4-5-7
9. 1-2-4-6-7	10. 1-2-5-6-7	11. 1-3-4-5-6	12. 1-3-4-5-7
13. 1-3-4-6-7	14. 1-3-5-6-7	15. 1-4-5-6-7	16. 2-3-4-5-6
17. 2-3-4-5-7	18. 2-3-4-6-7	19. 2-3-5-6-7	20. 2-4-5-6-7
21. 3-4-5-6-7	22. 0-0-0-0-0	23. 0-0-0-0-0	24. 0-0-0-0-0

C=7!/[5! (7 – 5)!] C=21

Six-color graphs:

1. 1-2-3-4-5-6	2. 1-2-3-4-5-7	3. 1-2-3-4-6-7	4.1-2-3-5-6-7
5. 1-2-4-5-6-7	6.1-3-4-5-6-7	7. 2-3-4-5-6-7	8. 0-0-0-0-0-0

C=7!/[6! (7 – 6)!] C=7

Seven-color graphs:

1. 1 - 2 - 3 - 4 - 5 - 6 -7

Citation: Karpushkin EV (2016) The ABC's of the Mathematical Infinitology. Principles of the Modern Theory and Practice of Scientific-and-Mathematical Infinitology. J Generalized Lie Theory Appl S2: 011. doi:10.4172/1736-4337.S2-011

C=7!/[7! (7 – 7)!] C=1

In a result of our elementary calculations with using the formulas, that well-known in combinatorics, we have received at last exactly 127 different compositions of the seven color-coded consequences of the natural numbers. Such a big quantity of combinations between the numbers and seven main colors allows to the makers of color illustrations "to draw" the natural mathematical Hermitage, consisting of the infinitely huge quantity of the scientific illustrations, borne by the theory of dot-color coding of the natural numbers on the immense spaces of the Cartesian or mathematical plus - minus infinity (± ∞: xy and xyz) (Mathematical rose in Figure 17).

Conclusion

Represented here in this article a new scientific method of graphical visualization of the natural numbers and consequences, forming by them, in view of chains of the colored dotes and sets in 2D Cartesian coordinates became possible, when the Author of this article salved the nonstandard mathematical task, having united the "Ulam's spiral" and own invention with the rectangular system of Cartesian coordinates. The bright and very impressive illustrations were appearing in a result, as if someone has correctly distributed the confetti on the surface of the magic field, and even their Inventor himself was surprised very much observe his "drawings". Looking at my graphs and plots, the thought was born that no one in the World can create such "pictures" but Mr. Benoit B. Mandelbrot, the famous American mathematician that used in his mathematical creativity the complex numbers, his own fantasy and the simplest IBC PC programmable media products as well. The results of Mandelbrot's work are known to everybody, but new graphs and plots made by me are known to nobody to my big regret.

Many centuries ago, the French scientist R. Descartes has invented the method of representation the suited information in view of mathematical lines, curves and the schematic diagrams in a symbol net, where two lines were crossing under the angle of 90° forming a zero-point as the beginning of this system. But the most interesting illustrations in this system, named letter in honor of R. Descartes, were appearing when the mathematicians dissolved graphically the equations and different functional dependences like $y=x^2$, $y=x^3$ and a lot of others. Now, almost four century later from the invention of Cartesian coordinates system, this great idea of the French academician has become the first media in many sciences for decision of different mathematical tasks that can now decide any educated person from the school pupils and ending the Nobel Prize laureates.

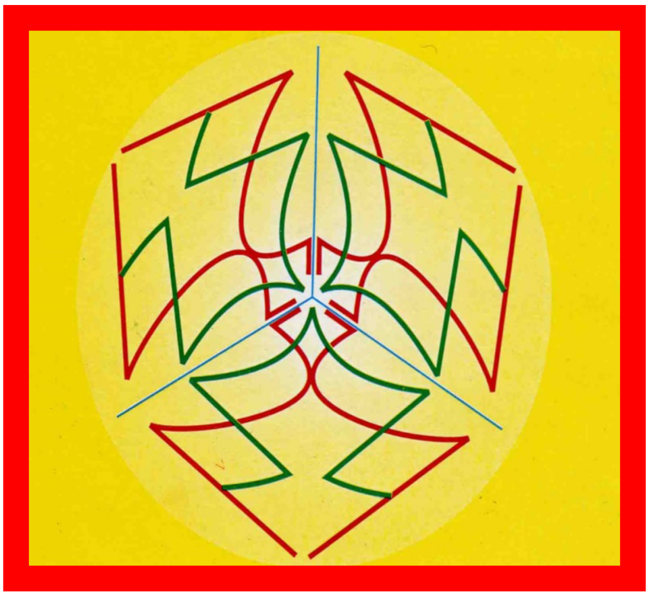


Figure 17: Mathematical rose.

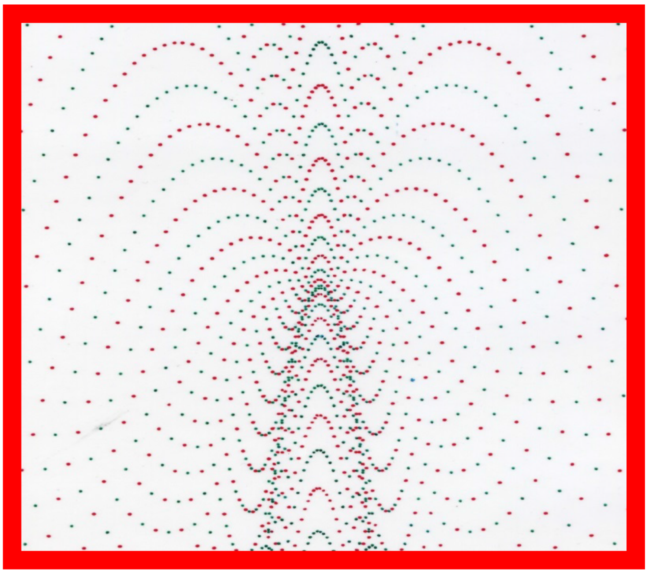
When the first natural numbers plots were created by me in the Cartesian coordinates, it has been noticed that the investigated idea has relation not only to a method of studying the natural numbers and their complex-algebraic equivalents but, how strange it may be, to the mathematical or Cartesian plus-minus infinity, the perfect theory of its studying and representing is worked by no one scientists up to this day. The graphic-and analytic method of visualization of natural numbers presented in this article opens widely the doors and gates for all and any persons, who will introduce with the main principles of this idea. And everything that it is needed for this work --- the elementary interest to this new idea in mathematics. Thanks to this method, one can make in the rectangular system of Cartesian coordinates some beautiful color dotted "photo portrait" of any natural number, for example, 1, 2, 3, 5, and 17. 35 etc., or the "picture" any, formed from them, consequence, such ones as the prime numbers, twin-numbers, Fibonacci numbers and etc.

In this article, special attention is paid to the specific rules and methods of calculation and creating the prime numbers graphs and other plots in Cartesian coordinates, having provided them preliminarily with a mathematical tables, where are listed all necessary information to create with their help the main mathematical "photos" of these consequences in the rectangular system of Cartesian coordinates. The method allows making the same illustrations in axonometric projection when the three axis are under the angle of 120° to one another. It is also existing the method of programming the Cartesian system with the help of the correspondence basic modules-stencils that can create the initial variant of the future colored - dotted mathematical illustrations that will allow to convert this idea into the huge interminable scientific kaleidoscope or mathematical casket with dozens of drawings and illustrations for further professional studying the natural numbers, their complex-algebraic equivalents themselves, and their colored graphics and the mathematical infinity as well.

List of Illustrations

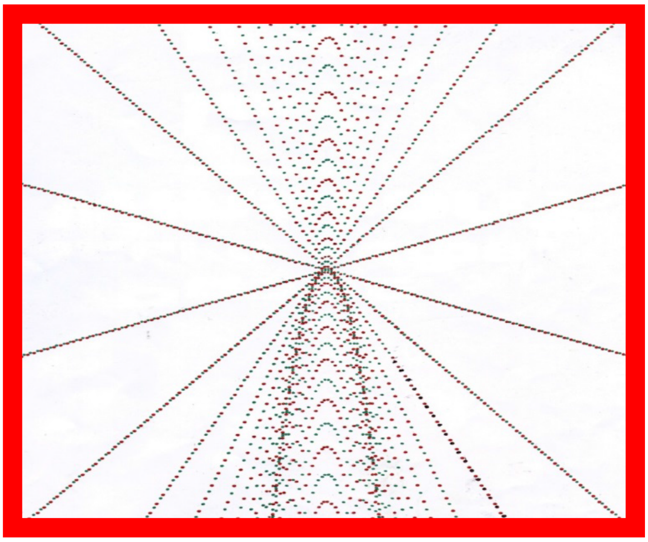
1. Fragment of the interminable red-green dotted plot of the Natural numbers consequence in Cartesian coordinates.
2. Fragment of the C++ - made interminable red dotted plot of the Natural prime numbers consequence in Cartesian coordinates.
3. Fragment of the interminable red-green dotted plot of the Natural – odd-even – numbers consequence in Cartesian coordinates.
4. Big Bang or four Black Holes merging (Fragment of the mathematical model).
5. Fragment of the interminable dark-blue dotted plot of the Natural twin numbers consequence in Cartesian coordinates (Figure 18-22).

List of Illustrations



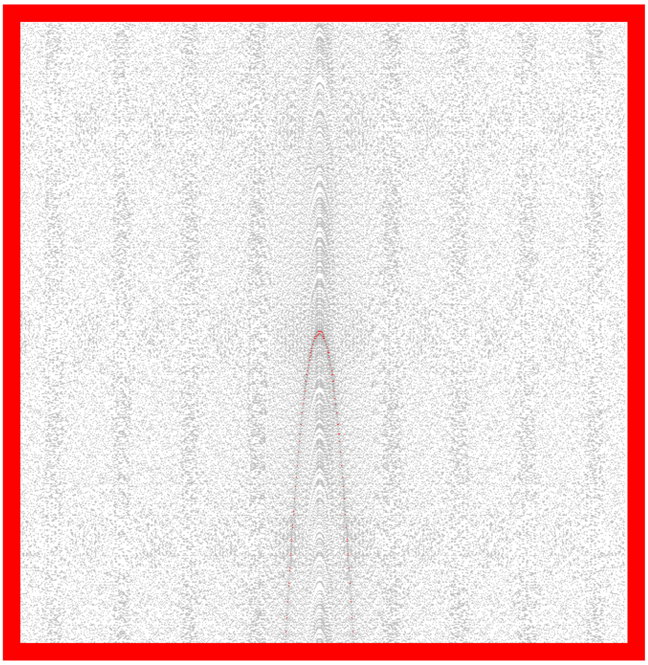
1. $\{A_n\} = \{n^2\}$

Figure 18: Fragment of the interminable red-green dotted plot of the Natural numbers consequence in Cartesian coordinates.



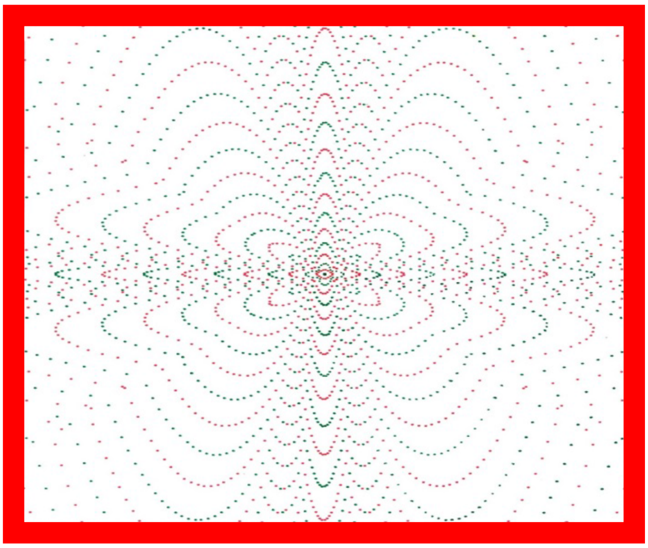
3. $\{A_n\} = \{(2n - 1)^2 \cup (4n^2)\}$

Figure 20: Fragment of the interminable red-green dotted plot of the Natural – odd-even – numbers consequence in Cartesian coordinates.



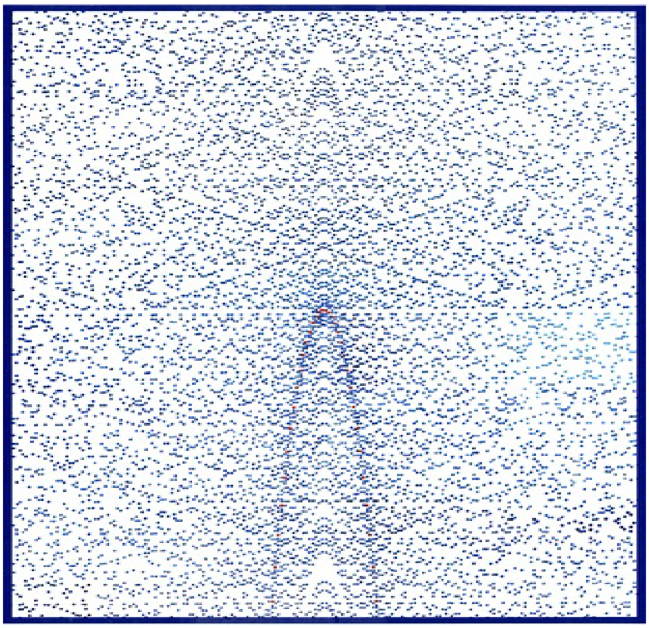
2. $\{A_n\} = \{\pi n\}$

Figure 19: Fragment of the interminable red dotted plot of the Natural prime numbers consequence in Cartesian coordinates.



4. Big Bang

Figure 21: Fragment of the interminable red-green dotted plot of the Natural numbers consequence view or Big Bang.



5. {An}={πn}

Figure 22: Fragment of the interminable dark-blue dotted plot of the Natural twin numbers consequence in Cartesian coordinates.

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Citation: Karpushkin EV (2016) The ABC's of the Mathematical Infinitology. Principles of the Modern Theory and Practice of Scientific-and-Mathematical Infinitology. J Generalized Lie Theory Appl S2: 011. doi:10.4172/1736-4337.S2-011

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, Unversity of Antananarivo, Madagascar

Journal of Generalized Lie Theory and Applications
ISSN: 1736-4337

Crooks, J Generalized Lie Theory Appl 2016, S2
http://dx.doi.org/10.4172/1736-4337.S2-012

Research Article Open Access

Properties of Nilpotent Orbit Complexification

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Abstract

We consider aspects of the relationship between nilpotent orbits in a semisimple real Lie algebra $\mathfrak{g}$ and those in its complexification $\mathfrak{g}_{\mathbb{C}}$. In particular, we prove that two distinct real nilpotent orbits lying in the same complex orbit are incomparable in the closure order. Secondly, we characterize those $\mathfrak{g}$ having non-empty intersections with all nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$. Finally, for $\mathfrak{g}$ quasi-split, we characterize those complex nilpotent orbits containing real ones.

Keywords: Nilpotent orbit; Quasi-split Lie algebra; Kostant-Sekiguchi correspondence

1. Introduction

1.1 Background and statement of results

Real and complex nilpotent orbits have received considerable attention in the literature. The former have been studied in a variety of contexts, including differential geometry, symplectic geometry, and Hodge theory [1]. Also, there has been some interest in concrete descriptions of the poset structure on real nilpotent orbits in specific cases [2,3]. By contrast, complex nilpotent orbits are studied in algebraic geometry [4,5,6] and representation theory — in particular, Springer Theory [7].

Attention has also been given to the interplay between real and complex nilpotent orbits, with the Kostant-Sekiguchi Correspondence being perhaps the most famous instance [8]. Accordingly, the present article provides additional points of comparison between real and complex nilpotent orbits. Specifically, let $\mathfrak{g}$ be a finite-dimensional semisimple real Lie algebra with complexification $\mathfrak{g}_{\mathbb{C}}$. Each real nilpotent orbit $\mathcal{O}\subseteq\mathfrak{g}$ lies in a unique complex nilpotent orbit $\mathcal{O}_{\mathbb{C}}\subseteq\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathcal{O}$. The following is our main result.

Theorem 1: The process of nilpotent orbit complexification has the following properties.

(i) Every complex nilpotent orbit is realizable as the complexification of a real nilpotent orbit if and only if $\mathfrak{g}$ is quasi-split and has no simple summand of the form $\mathfrak{so}(2n+1, 2n-1)$.

(ii) If $\mathfrak{g}$ is quasi-split, then a complex nilpotent orbit $\Theta\subseteq\mathfrak{g}_{\mathbb{C}}$ is realizable as the complexification of a real nilpotent orbit if and only if Θ is invariant under conjugation with respect to the real form $\mathfrak{g}\subseteq\mathfrak{g}_{\mathbb{C}}$.

(iii) If $\mathcal{O}_1,\mathcal{O}_2\subseteq\mathfrak{g}$ are real nilpotent orbits satisfying $(\mathcal{O}_1)_{\mathbb{C}}=(\mathcal{O}_2)_{\mathbb{C}}$, then either $\mathcal{O}_1=\mathcal{O}_2$ or these two orbits are incomparable in the closure order.

1.2 Structure of the article

We begin with an overview of nilpotent orbits in semisimple real and complex Lie algebras. In recognition of Theorem 1 (iii), and of the role played by the unique maximal complex nilpotent orbit $\Theta_{\text{reg}}(\mathfrak{g}_{\mathbb{C}})$ throughout the article, Section 2.2 reviews the closure orders on the sets of real and complex nilpotent orbits. In Section 2.3, we recall some of the details underlying the use of decorated partitions to index nilpotent orbits.

Section 3 is devoted to the proof of Theorem 1. In Section 3.1, we represent nilpotent orbit complexification as a poset map $\varphi_{\mathfrak{g}}$ between the collections of real and complex nilpotent orbits. Next, we show this map to have a convenient description in terms of decorated partitions. Section 3.2 then directly addresses the proof of Theorem 1 (i), formulated as a characterization of when $\varphi_{\mathfrak{g}}$ is surjective. Using Proposition 2, we reduce this exercise to one of characterizing surjectivity for $\mathfrak{g}$ simple. Together with the observation that surjectivity implies $\mathfrak{g}$ is quasi-split and is implied by $\mathfrak{g}$ being split, Proposition 2 allows us to complete the proof of Theorem 1 (i).

We proceed to Section 3.3, which provides the proof of Theorem 1 (ii). The essential ingredient is Kottwitz's work [9]. We also include Proposition 3, which gives an interesting sufficient condition for a complex nilpotent orbit to be in the image of $\varphi_{\mathfrak{g}}$.

In Section 3.4, we give a proof of Theorem 1 (iii). Our proof makes extensive use of the Kostant-Sekiguchi Correspondence, the relevant parts of which are mentioned.

2. Nilpotent Orbit Generalities

2.1 Nilpotent orbits

We begin by fixing some of the objects that will persist throughout this article. Let $\mathfrak{g}$ be a finite-dimensional semisimple real Lie algebra with adjoint group G. Also, let $\mathfrak{g}_{\mathbb{C}}:=\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ be the complexification of $\mathfrak{g}$, whose adjoint group is the complexification $G_{\mathbb{C}}$. One has the adjoint representations

$$Ad:G\to GL(\mathfrak{g}) \text{ and } Ad_{\mathbb{C}}:G_{\mathbb{C}}\to GL(\mathfrak{g}_{\mathbb{C}})$$

of G and $G_{\mathbb{C}}$, respectively. Differentiation then gives the adjoint representations of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$, namely

$$ad:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}) \text{ and } ad_{\mathbb{C}}:\mathfrak{g}_{\mathbb{C}}\to\mathfrak{gl}(\mathfrak{g}_{\mathbb{C}}).$$

Recall that an element $\xi\in\mathfrak{g}$ (resp. $\xi\in\mathfrak{g}_{\mathbb{C}}$) is called nilpotent if $ad(\xi):\mathfrak{g}\to\mathfrak{g}$ (resp. $ad_{\mathbb{C}}(\xi):\mathfrak{g}_{\mathbb{C}}\to\mathfrak{g}_{\mathbb{C}}$) is a nilpotent vector space endomorphism. The nilpotent cone $\mathcal{N}(\mathfrak{g})$ (resp. $\mathcal{N}(\mathfrak{g}_{\mathbb{C}})$) is then the subvariety of nilpotent elements of $\mathfrak{g}$ (resp. $\mathfrak{g}_{\mathbb{C}}$). A real (resp. complex) nilpotent orbit is an orbit of a nilpotent element in $\mathfrak{g}$ (resp. $\mathfrak{g}_{\mathbb{C}}$) under the adjoint representation of G (resp. $G_{\mathbb{C}}$). Since the adjoint representation occurs by means of Lie algebra automorphisms, a real (resp. complex)

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Received April 24, 2016; **Accepted** June 09, 2016; **Published** June 30, 2016

Citation: Crooks P (2016) Properties of Nilpotent Orbit Complexification. J Generalized Lie Theory Appl S2: 012. doi:10.4172/1736-4337.S2-012

nilpotent orbit is equivalently defined to be a G -orbit (resp. $G_{\mathbb{C}}$-orbit) in $\mathcal{N}(\mathfrak{g})$ (resp. $\mathcal{N}(\mathfrak{g}_{\mathbb{C}})$). By virtue of being an orbit of a smooth G -action, each real nilpotent orbit is an immersed submanifold of $\mathfrak{g}$. However, as $G_{\mathbb{C}}$ is a complex linear algebraic group, a complex nilpotent orbit is a smooth locally closed complex subvariety of $\mathfrak{g}_{\mathbb{C}}$.

2.2 The closure orders

The sets $\mathcal{N}(\mathfrak{g})/G$ and $\mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}}$ of real and complex nilpotent orbits are finite and carry the so-called closure order. In both cases, this is a partial order defined by

$$\mathcal{O}_1 \leq \mathcal{O}_2 \text{ if and only if } \mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}. \quad (1)$$

In the real case, one takes closures in the classical topology on $\mathfrak{g}$. For the complex case, note that a complex nilpotent orbit Θ is a constructible subset of $\mathfrak{g}_{\mathbb{C}}$, so that its Zariski and classical closures agree. Accordingly, $\overline{\Theta}$ shall denote this common closure.

Example 1: Suppose that $\mathfrak{g}_{\mathbb{C}}=\mathfrak{sl}_n(\mathbb{C})$, whose adjoint group is $G_{\mathbb{C}}=PSL_n(\mathbb{C})$. The nilpotent elements of $\mathfrak{sl}_n(\mathbb{C})$ are precisely the nilpotent $n \times n$ matrices, so that the nilpotent $PSL_n(\mathbb{C})$ -orbits are exactly the ($GL_n(\mathbb{C})$-) conjugacy classes of nilpotent matrices. The latter are indexed by the partitions of n via Jordan canonical forms. Given a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ of n, let Θ_λ be the $PSL_n(\mathbb{C})$-orbit of the nilpotent matrix with Jordan blocks of sizes $\lambda_1,\lambda_2,\ldots,\lambda_k$, read from top-to-bottom. It is a classical result of Gerstenhaber [10] that $\Theta_\lambda \leq \Theta_\mu$ if and only if $\lambda \leq \mu$ in the dominance order [11].

The poset $\mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}}$ has a unique maximal element $\Theta_{reg}(\mathfrak{g}_{\mathbb{C}})$, called the regular nilpotent orbit. It is the collection of all elements of $\mathfrak{g}_{\mathbb{C}}$ which are simultaneously regular and nilpotent. In the framework of Example 1, $\Theta_{reg}(\mathfrak{sl}_n(\mathbb{C}))$ corresponds to the partition(n).

2.3 Partitions of nilpotent orbits

Generalizing Example 1, it is often natural to associate a partition to each real and complex nilpotent orbit. One sometimes endows these partitions with certain decorations and then uses decorated partitions to enumerate nilpotent orbits. It will be advantageous for us to recall the construction of the underlying (undecorated) partitions. Our exposition will be largely based on Chapters 5 and 9 of [12].

Suppose that $\mathfrak{g}$ comes equipped with a faithful representation $\mathfrak{g} \subseteq \mathfrak{gl}(V)=End_{\mathbb{F}}(V)$, where V is a finite-dimensional vector space over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. The choice of V determines an assignment of partitions to nilpotent orbits in both $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$. To this end, fix a real nilpotent orbit $\mathcal{O} \subseteq \mathcal{N}(\mathfrak{g})$ and choose a point $\xi \in \mathcal{O}$. We may include ξ as the nilpositive element of an $\mathfrak{sl}_2(\mathbb{R})$ –triple (ξ,h,n), so that

$$[\xi,n]=h,[h,\xi]=2\xi,[h,n]=-2n$$

Regarding V as an $\mathfrak{sl}_2(\mathbb{R})$-module, one has a decomposition into irreducibles,

$$V=\bigoplus_{j=1}^{k} V_{\lambda_j},$$

where V_{λ_j} denotes the irreducible λ_j -dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$ over $\mathbb{F}$. Let us require that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, so that $(\lambda_1,\lambda_2,\ldots,\lambda_k)$ is a partition of $\dim_{\mathbb{F}}(V)$. Accordingly, we define the partition of $\mathcal{O}$ to be

$$\lambda(\mathcal{O}):=(\lambda_1,\lambda_2,\ldots,\lambda_k).$$

It can be established that $\lambda(\mathcal{O})$ depends only on $\mathcal{O}$.

The faithful representation V of $\mathfrak{g}$ canonically gives a faithful representation $\tilde{V}$ of $\mathfrak{g}_{\mathbb{C}}$. Indeed, if V is over $\mathbb{C}$, then one has an inclusion $\mathfrak{g}_{\mathbb{C}} \subseteq \mathfrak{gl}(V)$ (so $\tilde{V}=V$). If V is over $\mathbb{R}$, then the inclusion $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ complexifies to give a faithful representation $\mathfrak{g}_{\mathbb{C}} \subseteq \mathfrak{gl}(V_{\mathbb{C}})$ (ie. $\tilde{V}=V_{\mathbb{C}}$). In either case, one proceeds in analogy with the real nilpotent case, using the faithful representation to yield a partition $\lambda(\Theta)$ of a complex nilpotent orbit $\Theta \subseteq \mathcal{N}(\mathfrak{g})$. The only notable difference with the real case is that $\mathfrak{sl}_2(\mathbb{R})$ is replaced with $\mathfrak{sl}_2(\mathbb{C})$.

Example 2: One can use the framework developed above to index the nilpotent orbits in $\mathfrak{sl}_2(\mathbb{C})$ using the partitions of n. This coincides with the indexing given in Example 1.

Example 3: The nilpotent orbits in $\mathfrak{sl}_2(\mathbb{R})$ are indexed by the partitions of n, after one replaces certain partitions with decorated counterparts. Indeed, if λ is a partition of n having only even parts, we replace λ with the decorated partitions λ_+ and λ_-. Otherwise, we leave λ undecorated.

Example 4: Suppose that $n \geq 3$ and consider $\mathfrak{g}=\mathfrak{su}(p,q)$ with $1 \leq q \leq p$ and $p+q=n$. This Lie algebra is a real form of $\mathfrak{sl}_n(\mathbb{C})$. Now, let us regard a partition of n as a Young diagram with n boxes. Furthermore, recall that a signed Young diagram is a Young diagram whose boxes are marked with + or −, such that the signs alternate across each row [12]. We restrict our attention to the signed Young diagrams of signature (p,q), namely those for which + and − appear with respective multiplicities p and q. It turns out that the nilpotent orbits in $\mathfrak{su}(p,q)$ are indexed by the signed Young diagrams of signature (p,q).

Example 5: Suppose that $\mathfrak{g}_{\mathbb{C}}=\mathfrak{so}_{2n}(\mathbb{C})$ with $n \geq 4$. Taking our faithful representation to be $\mathbb{C}^{2n}$, nilpotent orbits in $\mathfrak{so}_{2n}(\mathbb{C})$ are assigned partitions of $2n$. The partitions realized in this way are those in which each even part appears with even multiplicity. One extends these partitions to an indexing set by replacing each λ having only even parts with the decorated partitions λ_+ and λ_-.

Example 6: Suppose that $n \geq 3$ and consider $\mathfrak{g}_{\mathbb{C}}=\mathfrak{so}(p,q)$ with $1 \leq q \leq p$ and $p+q=n$. Note that $\mathfrak{so}(p,q)$ is a real form of $\mathfrak{g}_{\mathbb{C}}=\mathfrak{so}_n(\mathbb{C})$. As with Example 4, we will identify partitions of n with Young diagrams having n boxes. We begin with the signed Young diagrams of signature (p,q) such that each even-length row appears with even multiplicity and has its leftmost box marked with +. To obtain an indexing set for the nilpotent orbits in $\mathfrak{so}(p,q)$, we decorate two classes of these signed Young diagrams Y. Accordingly, if Y has only even-length rows, then remove Y and add the four decorated diagrams $Y_{+,+}, Y_{+,-}, Y_{-,+}$ and $Y_{-,-}$. Secondly, suppose that Y has at least one odd-length row, and that each such row has an even number of boxes marked +, or that each such row has an even number of boxes marked −. In this case, we remove Y and add the decorated diagrams Y_+ and Y.

3. Nilpotent Orbit Complexification

3.1 The complexification map

There is a natural way in which a real nilpotent orbit determines a complex one. Indeed, the inclusion $\mathcal{N}(\mathfrak{g}) \subseteq \mathcal{N}(\mathfrak{g}_{\mathbb{C}})$ gives rise to a map.

$$\varphi_{\mathfrak{g}}:\mathcal{N}(\mathfrak{g})/G \to \mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}}$$

$$\mathcal{O} \mapsto \mathcal{O}_{\mathbb{C}}.$$

Concretely, $\mathcal{O}_{\mathbb{C}}$ is just the unique complex nilpotent orbit containing $\mathcal{O}$, and we shall call it the complexification of $\mathcal{O}$. Let us then call $\varphi_{\mathfrak{g}}$ the complexification map for $\mathfrak{g}$.

It will be prudent to note that the process of nilpotent orbit complexification is well-behaved with respect to taking partitions. More explicitly, we have the following proposition.

Proposition 1: Suppose that $\mathfrak{g}$ is endowed with a faithful representation $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. If $\mathcal{O}$ is a real nilpotent orbit, then $\lambda(\mathcal{O}_{\mathbb{C}})=\lambda(\mathcal{O})$.

Proof: Choose a point $\xi \in \mathcal{O}$ and include it in an $\mathfrak{sl}_2(\mathbb{R})$-triple (ξ,h,η) as in Section 2.3. Note that (ξ,h,η) is then additionally an $\mathfrak{sl}_2(\mathbb{C})$-triple in $\mathfrak{g}_{\mathbb{C}}$. Hence, we will prove that the faithful representation $\tilde{V}$ of $\mathfrak{g}_{\mathbb{C}}$ decomposes into irreducible $\mathfrak{sl}_2(\mathbb{C})$-representations according to the partition $\lambda(\mathcal{O})$.

Let us write $\lambda(\mathcal{O})=(\lambda_1,\ldots,\lambda_k)$, so that

$$V = \bigoplus_{j=1}^{k} V_{\lambda_j} \tag{2}$$

is the decomposition of V into irreducible $\mathfrak{sl}_2(\mathbb{R})$-representations. If V is over $\mathbb{C}$, then $\tilde{V}=V$ and (2) is a decomposition of $\tilde{V}$ into irreducible $\mathfrak{sl}_2(\mathbb{C})$-representations. If V is over $\mathbb{R}$, then $\tilde{V}=V_{\mathbb{C}}$ and

$$V_{\mathbb{C}} = \bigoplus_{j=1}^{k} (V_{\lambda_j})_{\mathbb{C}}$$

is the decomposition of $\tilde{V}$ into irreducible representations of $\mathrm{sl}_2(\mathbb{C})$. In each of these two cases, we have $\lambda(\mathcal{O}_{\mathbb{C}})=\lambda(\mathcal{O})$.

Proposition 1 allows us to describe $\varphi_{\mathfrak{g}}$ in more combinatorial terms. To this end, fix a faithful representation $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. As in Examples 2-6, we obtain index sets $I(\mathfrak{g})$ and $I(\mathfrak{g}_{\mathbb{C}})$ of decorated partitions for the real and complex nilpotent orbits, respectively. We may therefore regard $\varphi_{\mathfrak{g}}$ as a map

$\varphi_{\mathfrak{g}}: I(\mathfrak{g}) \to I(\mathfrak{g}_{\mathbb{C}})$.

Now, let $P(\mathfrak{g}_{\mathbb{C}})$ be the set of all partitions of the form $\lambda(\Theta)$, with $\Theta \subseteq \mathfrak{g}_{\mathbb{C}}$ a complex nilpotent orbit. One has the map

$I(\mathfrak{g}) \to P(\mathfrak{g}_{\mathbb{C}})$,

sending a decorated partition to its underlying partition. Proposition 1 is then the statement that the composite map

$$I(\mathfrak{g}) \xrightarrow{\varphi_{\mathfrak{g}}} I(\mathfrak{g}_{\mathbb{C}}) \to P(\mathfrak{g}_{\mathbb{C}})$$

sends an index in $I(\mathfrak{g})$ to its underlying partition. Let us denote this composite map by $\psi_{\mathfrak{g}}: I(\mathfrak{g}) \to P(\mathfrak{g}_{\mathbb{C}})$.

We will later give a characterization of those semisimple real Lie algebras $\mathfrak{g}$ for which $\varphi_{\mathfrak{g}}$ is surjective. To help motivate this, we investigate the matter of surjectivity in some concrete examples.

Example 7: Recall the parametrizations of nilpotent orbits in $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{g}_{\mathbb{C}}=\mathfrak{sl}_2(\mathbb{C})$ outlined in Examples 3 and 2, respectively. We see that $I(\mathfrak{g})=P(\mathfrak{g}_{\mathbb{C}})$ and $\varphi_{\mathfrak{g}}=\psi_{\mathfrak{g}}$. The surjectivity of $\varphi_{\mathfrak{g}}$ then follows immediately from that of $\psi_{\mathfrak{g}}$.

Example 8: Let the nilpotent orbits in $\mathfrak{g}=\mathfrak{su}(n,n)$ be parametrized as in Example 4. We then have $\mathfrak{g}_{\mathbb{C}}=\mathfrak{sl}_{2n}(\mathbb{C})$, whose nilpotent orbits are indexed by the partitions of $2n$. Given such a partition λ, let Y denote the corresponding Young diagram. Since Y has an even number of boxes, it has an even number, $2k$, of odd-length rows. Label the leftmost box in k of these rows with +, and label the leftmost box in each of the remaining k rows with −. Now, complete this labelling to obtain a signed Young diagram $\tilde{Y}$, noting that $\tilde{Y}$ then has signature (n,n). Hence, $\tilde{Y}$ corresponds to a nilpotent orbit in $\mathrm{su}(n,n)$ and $\psi_{\mathfrak{g}}(\tilde{Y})=\lambda$. It follows that $\psi_{\mathfrak{g}}$ is surjective. Since $I(\mathfrak{g})=P(\mathfrak{g}_{\mathbb{C}})$ and $\varphi_{\mathfrak{g}}=\psi_{\mathfrak{g}}$, we have shown $\varphi_{\mathfrak{g}}$ to be surjective. A similar argument establishes surjectivity when $\mathfrak{g}=\mathfrak{su}(n+1,n)$.

Example 9: Let us consider $\mathfrak{g}=\mathfrak{so}(2n+2,2n)$, with nilpotent orbits indexed as in Example 6. Noting Example 5, a partition λ of $4n+2$ represents a nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}=\mathfrak{so}_{4n+2}(\mathbb{C})$ if and only if each even part of λ occurs with even multiplicity. Since $4n+2$ is even and not divisible by 4, it follows that any such λ has exactly $2k$ odd parts for some $k \geq 1$. Let Y be the Young diagram corresponding to λ, and label the leftmost box in $k-1$ of the odd-length rows with +. Next, label the leftmost box in each of $k-1$ different odd-length rows with −. Finally, use + to label the leftmost box in each of the two remaining odd-length rows. Let $\tilde{Y}$ be any completion of our labelling to a signed Young diagram, such that the leftmost box in each even-length row is marked with +. Note that $\tilde{Y}$ has signature $(2n+2,2n)$. It follows that $\tilde{Y}$ represents a nilpotent orbit in $\mathfrak{so}(2n+2,2n)$ and $\psi_{\mathfrak{g}}(\tilde{Y})=\lambda$. Furthermore, $I(\mathfrak{g}_{\mathbb{C}})=P(\mathfrak{g}_{\mathbb{C}})$ and $\varphi_{\mathfrak{g}}=\psi_{\mathfrak{g}}$, so that $\varphi_{\mathfrak{g}}$ is surjective.

Example 10: Suppose that $\mathfrak{g}=\mathfrak{so}(2n+2,2n-1)$, whose nilpotent orbits are parametrized in Example 6. Let the nilpotent orbits in $\mathfrak{g}=\mathfrak{so}_{4n}(\mathbb{C})$ be indexed as in Example 5. There exist partitions of $4n$ having only even parts, with each part appearing an even number of times. Let λ be one such partition, which by Example 6 represents a nilpotent orbit in $\mathfrak{so}_{4n}(\mathbb{C})$. Note that every signed Young diagram with underlying partition λ must have signature $(2n,2n)$. In particular, λ cannot be realized as the image under $\psi_{\mathfrak{g}}$ of a signed Young diagram indexing a nilpotent orbit in $\mathfrak{so}(2n+2,2n-1)$. It follows that $\psi_{\mathfrak{g}}$ and $\varphi_{\mathfrak{g}}$ are not surjective.

3.2 Surjectivity

We now address the matter of classifying those semisimple real Lie algebras $\mathfrak{g}$ for which $\varphi_{\mathfrak{g}}$ is surjective. To proceed, we will require some additional machinery. Let $\mathfrak{p} \subseteq \mathfrak{g}$ be the (-1)-eigenspace of a Cartan involution, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Also, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$, and choose a fundamental Weyl chamber $C \subseteq \mathfrak{h}$. Given a complex nilpotent orbit $\Theta \subseteq \mathfrak{g}_{\mathbb{C}}$, there exists an $\mathfrak{sl}_2(\mathbb{C})$-triple (ξ,h,η) in $\mathfrak{g}_{\mathbb{C}}$ with the property that $\xi \in \Theta$ and $h \in C$. The element $h \in C$ is uniquely determined by this property, and is called the characteristic of Θ.

Theorem 1 of [13] then states that $\Theta\mathfrak{g} \neq \emptyset$ if and only if $h \in \mathfrak{a}$. If $\mathfrak{g}$ is split, then $\mathfrak{a}=\mathfrak{h}$, and the following lemma is immediate.

Lemma 1: If $\mathfrak{g}$ is split, then $\varphi_{\mathfrak{g}}$ is surjective.

Let us now consider necessary conditions for surjectivity. To this end, recall that $\mathfrak{g}$ is called quasi-split if there exists a subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ such that $\mathfrak{b}_{\mathbb{C}}$ is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$. However, the following characterization of being quasi-split will be more suitable for our purposes.

Lemma 2: The Lie algebra $\mathfrak{g}$ is quasi-split if and only if $\Theta_{\mathrm{reg}}(\mathfrak{g}_{\mathbb{C}})$ is in the image of $\varphi_{\mathfrak{g}}$. In particular, $\mathfrak{g}$ being quasi-split is a necessary condition for $\varphi_{\mathfrak{g}}$ to be surjective.

Proof: Proposition 5.1 of [14], states that $\mathfrak{g}$ is quasi-split if and only if $\mathfrak{g}$ contains a regular nilpotent element of $\mathfrak{g}_{\mathbb{C}}$. Since $\Theta_{\mathrm{reg}}(\mathfrak{g}_{\mathbb{C}})$ consists of all such elements, this is equivalent to having $\Theta_{\mathrm{reg}}(\mathfrak{g}_{\mathbb{C}}) \cap \mathfrak{g} \neq \emptyset$ hold. This latter condition holds precisely when $\Theta_{\mathrm{reg}}(\mathfrak{g}_{\mathbb{C}})$ is in the image of $\varphi_{\mathfrak{g}}$.

Lemmas 1 and 2 establish that $\varphi_{\mathfrak{g}}$ being surjective is a weaker condition than having $\mathfrak{g}$ be split, but stronger than having $\mathfrak{g}$ be quasi-split. Furthermore, since $\mathfrak{su}(n,n)$ is not a split real form of $\mathfrak{sl}_{2n}(\mathbb{C})$, Example 8 establishes that surjectivity is strictly weaker than $\mathfrak{g}$ being split. Yet, as $\mathfrak{so}(2n+2,2n-1)$ is a quasi-split real form of $\mathfrak{so}_{4n}(\mathbb{C})$, Example 10 demonstrates that surjectivity is strictly stronger than having $\mathfrak{g}$ be quasi-split. To obtain a more precise measure of the strength of the surjectivity condition, we will require the following proposition.

Proposition 2: Suppose that $\mathfrak{g}$ decomposes as a Lie algebra into

$$\mathfrak{g} = \bigoplus_{j=1}^{k} \mathfrak{g}_j,$$

Where $\mathfrak{g}_1,\ldots,\mathfrak{g}_k$ are simple real Lie algebras. Let $G_1,\ldots,G_k$ denote the respective adjoint groups.

(i) The map $\varphi_\mathfrak{g}: \mathcal{N}(\mathfrak{g})/G \to \mathcal{N}(\mathfrak{g}_\mathbb{C})/G_\mathbb{C}$ is surjective if and only each orbit complexification map $\varphi_{\mathfrak{g}_j}: \mathcal{N}(\mathfrak{g}_j)/G_j \to \mathcal{N}((\mathfrak{g}_j)_\mathbb{C})/(G_j)_\mathbb{C}$ is surjective.

(ii) The Lie algebra $\mathfrak{g}$ is quasi-split if and only if each summand $\mathfrak{g}_j$ is quasi-split.

Proof: For each $j\in\{1,\ldots,k\}$, let $\pi_j:\mathfrak{g}\to\mathfrak{g}_j$ be the projection map. Note that $\xi \in \mathfrak{g}$ is nilpotent if and only if $\pi_j(\xi)$ is nilpotent in $\mathfrak{g}_j$ for each j. It follows that

$$\pi : \mathcal{N}(\mathfrak{g}) \to \prod_{j=1}^{k} \mathcal{N}(\mathfrak{g}_j)$$
$$\xi \mapsto (\pi_j(\xi))_{j=1}^{k}$$

defines an isomorphism of real varieties. Note that $G = \prod_{j=1}^{k} G_j$, with the former group acting on $\mathcal{N}(\mathfrak{g})$ and the latter group acting on the product of nilpotent cones.

One then sees that π is G-equivariant, so that it descends to a bijection

$$\overline{\pi} : \mathcal{N}(\mathfrak{g})/G \to \prod_{j=1}^{k} \mathcal{N}(\mathfrak{g}_j)/G_j.$$

Analogous considerations give a second bijection

$$\overline{\pi}_\mathbb{C} : \mathcal{N}(\mathfrak{g}_\mathbb{C})/G_\mathbb{C} \to \prod_{j=1}^{k} \mathcal{N}((\mathfrak{g}_j)_\mathbb{C})/(G_j)_\mathbb{C}.$$

Furthermore, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{N}(\mathfrak{g})/G & \xrightarrow{\overline{\pi}} & \prod_{j=1}^{k} \mathcal{N}(\mathfrak{g}_j)/G_j \\ \downarrow \varphi_\mathfrak{g} & & \downarrow \prod_{j=1}^{k} \varphi_{\mathfrak{g}_j} \\ \mathcal{N}(\mathfrak{g}_\mathbb{C})/G_\mathbb{C} & \xrightarrow{\overline{\pi}_\mathbb{C}} & \prod_{j=1}^{k} \mathcal{N}((\mathfrak{g}_j)_\mathbb{C})/(G_j)_\mathbb{C} \end{array} \quad (3)$$

Hence, $\varphi_\mathfrak{g}$ is surjective if and only if $\prod_{j=1}^{k} \varphi_{\mathfrak{g}_j}$ is so, proving (i).

By Lemma 2, proving (ii) will be equivalent to proving that $\Theta_{\text{reg}}(\mathfrak{g}_\mathbb{C})$ is in the image of $\varphi_\mathfrak{g}$ if and only if $\Theta_{\text{reg}}((\mathfrak{g}_j))$ is in the image of $\varphi_{\mathfrak{g}_j}$ for all j. Using the diagram (3), this will follow from our proving that the image of $\Theta_{\text{reg}}(\mathfrak{g}_\mathbb{C})$ under $\overline{\pi}_\mathbb{C}$ is the k-tuple of the regular nilpotent orbits in the $(\mathfrak{g}_j)_\mathbb{C}$, namely that

$$\overline{\pi}_\mathbb{C}(\Theta_{\text{reg}}(\mathfrak{g}_\mathbb{C})) = (\Theta_{\text{reg}}((\mathfrak{g}_j)_\mathbb{C}))_{j=1}^{k}. \quad (4)$$

To see this, note that $\prod_{j=1}^{k} \Theta_{\text{reg}}((\mathfrak{g}_j)_\mathbb{C})$ is the $G_\mathbb{C} = \prod_{j=1}^{k} (G_j)_\mathbb{C}$ -orbit of maximal dimension in $\prod_{j=1}^{k} \mathcal{N}((\mathfrak{g}_j)_\mathbb{C})$. This orbit is therefore the image of $\Theta_{\text{reg}}(\mathfrak{g}_\mathbb{C})$ under the $G_\mathbb{C}$-equivariant variety isomorphism $\mathcal{N}(\mathfrak{g}_\mathbb{C}) \cong \prod_{j=1}^{k} \mathcal{N}((\mathfrak{g}_j)_\mathbb{C})$, implying that (4) holds.

In light of Proposition 2, we address ourselves to classifying the simple real Lie algebras $\mathfrak{g}$ with surjective orbit complexification maps $\varphi_\mathfrak{g}$. Noting Lemma 2, we may assume $\mathfrak{g}$ to be quasi-split. Since $\mathfrak{g}$ being split is a sufficient condition for surjectivity, we are further reduced to finding those quasi-split simple $\mathfrak{g}$ which are non-split but have surjective $\varphi_\mathfrak{g}$. It follows that $\mathfrak{g}$ belongs to one of the four families $\mathfrak{su}(n,n)$, $\mathfrak{su}(n+1,n)$, $\mathfrak{so}(2n+2,2n)$, and $\mathfrak{so}(2n+1,2n-1)$, or that $\mathfrak{g}=EII$, the non-split, quasi-split real form of E_6 [15]. Our examples establish that $\varphi_\mathfrak{g}$ is surjective for $\mathfrak{g}=\mathfrak{su}(n,n)$, $\mathfrak{g}=\mathfrak{su}(n+1,n)$, and $\mathfrak{g}=\mathfrak{so}(2n+2,2n)$, while Example 10 demonstrates that surjectivity does not hold for $\mathfrak{g}=\mathfrak{so}(2n+1,2n-1)$. Also, a brief examination of the computations in [3] reveals that $\varphi_\mathfrak{g}$ is surjective for $\mathfrak{g}=EII$. We then have the following characterization of the surjectivity condition.

Theorem 2: If $\mathfrak{g}$ is a semisimple real Lie algebra, then $\varphi_\mathfrak{g}$ is surjective if and only if $\mathfrak{g}$ is quasi-split and has no simple summand of the form $\mathfrak{so}(2n+1,2n-1)$.

Proof: If $\varphi_\mathfrak{g}$ is surjective, then Lemma 2 implies that $\mathfrak{g}$ is quasi-split. Also, Proposition 2 implies that each simple summand of $\mathfrak{g}$ has a surjective orbit complexification map, and the above discussion then establishes that $\mathfrak{g}$ has no simple summand of the form $\mathfrak{so}(2n+1,2n-1)$. Conversely, assume that $\mathfrak{g}$ is quasi-split and has no simple summand of the form $\mathfrak{so}(2n+1,2n-1)$. By Proposition 2 (ii), each simple summand of $\mathfrak{g}$ is quasi-split. Furthermore, the above discussion implies that the only quasi-split simple real Lie algebras with non-surjective orbit complexification maps are those of the form $\mathfrak{so}(2n+1,2n-1)$. Hence, each simple summand of $\mathfrak{g}$ has a surjective orbit complexification map, and Proposition 2 (i) implies that $\varphi_\mathfrak{g}$ is surjective.

3.3 The Image of $\varphi_\mathfrak{g}$

Having investigated the surjectivity of $\varphi_\mathfrak{g}$, let us consider the more subtle matter of characterizing its image. Accordingly, let $\sigma_\mathfrak{g}:\mathfrak{g}_\mathbb{C}\to\mathfrak{g}_\mathbb{C}$ denote complex conjugation with respect to the real form $\mathfrak{g}\subseteq\mathfrak{g}_\mathbb{C}$. The following lemma will be useful.

Lemma 3: If $\Theta\subseteq\mathfrak{g}_\mathbb{C}$ is a complex nilpotent orbit, then so is $\sigma_\mathfrak{g}(\Theta)$.

Proof: Note that $\sigma_\mathfrak{g}$ integrates to a real Lie group automorphism

$\tau:(G_\mathbb{C})_{SC}\to(G_\mathbb{C})_{SC}$,

where $(G_\mathbb{C})_{SC}$ is the connected, simply-connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$. If $\mathfrak{g}\in (G_\mathbb{C})_{SC}$ and $\xi\in\mathfrak{g}_\mathbb{C}$, then

$\sigma_\mathfrak{g}(Ad(\mathfrak{g})(\xi))=Ad(\tau(\mathfrak{g}))(\sigma_\mathfrak{g}(\xi))$.

Hence, $\sigma_\mathfrak{g}$ sends the $(G)_{SC}$ -orbit of ξ to the $(G_\mathbb{C})_{SC}$ -orbit of $\sigma_\mathfrak{g}(\xi)$. To complete the proof, we need only observe that $(G_\mathbb{C})_{SC}$ -orbits coincide with $G_\mathbb{C}$-orbits in $\mathfrak{g}_\mathbb{C}$, and that $\sigma_\mathfrak{g}(\xi)$ is nilpotent whenever ξ is nilpotent.

We may now use $\sigma_\mathfrak{g}$ to explicitly describe the image of $\varphi_\mathfrak{g}$ when $\mathfrak{g}$ is quasi-split.

Theorem 3: If Θ is a complex nilpotent orbit, the condition $\sigma_\mathfrak{g}(\Theta)=\Theta$ is necessary for Θ to be in the image of $\varphi_\mathfrak{g}$. If $\mathfrak{g}$ is quasi-split, then this condition is also sufficient.

Proof: Assume that Θ belongs to the image of $\varphi_\mathfrak{g}$, so that there exists $\xi\in\Theta\cap\mathfrak{g}$. Note that $\sigma_\mathfrak{g}(\Theta)$ is then the complex nilpotent orbit containing $\sigma_\mathfrak{g}(\xi)=\xi$, meaning that $\sigma_\mathfrak{g}(\Theta)=\Theta$. Conversely, assume that g is quasi-split and that $\sigma_\mathfrak{g}(\Theta)=\Theta$. The latter means precisely that Θ is defined over $\mathbb{R}$ with respect to the real structure on $\mathfrak{g}_\mathbb{C}$ induced by the inclusion $\mathfrak{g}\subseteq\mathfrak{g}_\mathbb{C}$. Theorem 4.2 of [9] then implies that $\Theta\cap\mathfrak{g}\neq\emptyset$.

Using Theorem 3, we will give an interesting sufficient condition for a complex nilpotent orbit to be in the image of $\varphi_\mathfrak{g}$ when $\mathfrak{g}$ is quasi-split. In order to proceed, however, we will need a better understanding of the way in which $\sigma_\mathfrak{g}$ permutes complex nilpotent orbits. To this end, we have the following lemma.

Lemma 4: Suppose that $\mathfrak{g}$ comes with the faithful representation $\mathfrak{g}\subseteq\mathfrak{gl}(V)$, where V is over $\mathbb{R}$. If Θ is a complex nilpotent orbit, then $\lambda(\sigma_\mathfrak{g}(\Theta))=\lambda(\Theta)$.

Citation: Crooks P (2016) Properties of Nilpotent Orbit Complexification. J Generalized Lie Theory Appl S2: 012. doi:10.4172/1736-4337.S2-012

Proof: Choose an $\mathfrak{sl}_2(\mathbb{C})$-triple (ξ,h,η) in $\mathfrak{g}$ with $\xi\in\Theta$. Since $\sigma_{\mathfrak{g}}$ preserves Lie brackets, it follows that $(\sigma_{\mathfrak{g}}(\xi),\sigma_{\mathfrak{g}}(h),\sigma_{\mathfrak{g}}(\eta))$ is also an $\mathfrak{sl}_2(\mathbb{C})$-triple. The exercise is then to show that our two $\mathfrak{sl}_2(\mathbb{C})$-triples give isomorphic representations of $\mathfrak{sl}_2(\mathbb{C})$ on $\tilde{V}=V_{\mathbb{C}}$. For this, it will suffice to prove that h and $\sigma_{\mathfrak{g}}(h)$ act on $V_{\mathbb{C}}$ with the same eigenvalues, and that their respective eigenspaces for a given eigenvalue are equi-dimensional. To this end, let $\sigma_V:V_{\mathbb{C}}\to V_{\mathbb{C}}$ be complex conjugation with respect to $V\subseteq V_{\mathbb{C}}$. Note that

$$\sigma_{\mathfrak{g}}(h).(\sigma_V(x))=\sigma_V(h.x)$$

for all $x\in V_{\mathbb{C}}$, where . is used to denote the action of $\mathfrak{g}_{\mathbb{C}}$ on $V_{\mathbb{C}}$. Hence, if x is an eigenvector of h with eigenvalue $\lambda\in\mathbb{R}$, then $\sigma_V(x)$ is an eigenvector of $\sigma_{\mathfrak{g}}(h)$ with eigenvalue λ. We conclude that h and $\sigma_{\mathfrak{g}}(h)$ have the same eigenvalues. Furthermore, their respective eigenspaces for a fixed eigenvalue are related by σ_V, and so are equi-dimensional.

We now have the following

Proposition 3: Let $\mathfrak{g}$ be a quasi-split semisimple real Lie algebra endowed with a faithful representation $\mathfrak{g}\subseteq\mathfrak{gl}(V)$, where V is over $\mathbb{R}$. If Θ is the unique complex nilpotent orbit with partition $\lambda(\Theta)$, then Θ is in the image of $\varphi_{\mathfrak{g}}$.

Proof: By Lemma 4, $\sigma_{\mathfrak{g}}(\Theta)$ is a complex nilpotent orbit with partition $\lambda(\Theta)$, and our hypothesis on Θ gives $\sigma_{\mathfrak{g}}(\Theta)=\Theta$. Theorem 3 then implies that Θ is in the image of $\varphi_{\mathfrak{g}}$.

A few remarks are in order.

Remark 1: One can use Proposition 3 to investigate whether $\varphi_{\mathfrak{g}}$ is surjective without appealing to the partition-type description of $\varphi_{\mathfrak{g}}$ discussed in Section 3.1. For instance, suppose that $\mathfrak{g}=\mathfrak{so}(2n+2,2n)$, a quasi-split real form of $\mathfrak{g}_{\mathbb{C}}=\mathfrak{so}_{4n+2}(\mathbb{C})$. We refer the reader to Example 5 for the precise assignment of partitions to nilpotent orbits in $\mathfrak{so}_{4n+2}(\mathbb{C})$. In particular, note that a complex nilpotent orbit is the unique one with its partition if and only if the partition does not have all even parts. Furthermore, as discussed in Example 9, there do not exist partitions of $4n+2$ having only even parts such that each part appears with even multiplicity. Hence, each complex nilpotent orbit is specified by its partition, so Proposition 3 implies that $\varphi_{\mathfrak{g}}$ is surjective.

Remark 2: The converse of Proposition 3 does not hold. Indeed, suppose that $\mathfrak{g}_{\mathbb{C}}=\mathfrak{so}(2n,2n)$, the split real form of $\mathfrak{g}_{\mathbb{C}}=\mathfrak{so}_{4n}(\mathbb{C})$. Recalling Example 5, every partition of $4n$ with only even parts, each appearing with even multiplicity, is the partition of two distinct complex nilpotent orbits. Yet, Lemma 1 implies that $\varphi_{\mathfrak{g}}$ is surjective, so that these orbits are in the image of $\varphi_{\mathfrak{g}}$.

3.4 Fibres

In this section, we investigate the fibres of the orbit complexification map $\varphi_{\mathfrak{g}}:\mathcal{N}(\mathfrak{g})/G\to\mathcal{N}(\mathfrak{g}_{\mathbb{C}})/G_{\mathbb{C}}$. In order to proceed, it will be necessary to recall some aspects of the Kostant-Sekiguchi Correspondence. To this end, fix a Cartan involution $\theta:\mathfrak{g}\to\mathfrak{g}$. Letting $\mathfrak{k}$ and $\mathfrak{p}$ denote the 1 and (-1)-eigenspaces of θ, respectively, we obtain the internal direct sum decomposition

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}.$$

This gives a second decomposition

$$\mathfrak{g}_{\mathbb{C}}=\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{p}_{\mathbb{C}},$$

where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are the complexifications of $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Let $K\subseteq G$ and $K_{\mathbb{C}}\subseteq G_{\mathbb{C}}$ be the connected closed subgroups with respective Lie algebras $\mathfrak{k}$ and $\mathfrak{k}_{\mathbb{C}}$. The Kostant-Sekiguchi Correspondence is one between the nilpotent orbits in g and the $K_{\mathbb{C}}$-orbits in the ($K_{\mathbb{C}}$-invariant) subvariety $\mathfrak{p}_{\mathbb{C}}\cap\mathcal{N}(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$.

Theorem 4: (The Kostant-Sekiguchi Correspondence) There is a bijective correspondence

$$\mathcal{N}(\mathfrak{g})/G\to(\mathfrak{p}\cap\mathcal{N}(\mathfrak{g}_{\mathbb{C}}))/K_{\mathbb{C}}$$

$$\mathcal{O}\mapsto\mathcal{O}^{\vee}$$

with the following properties.

(i) It is an isomorphism of posets, where $(\mathfrak{p}_{\mathbb{C}}\cap\mathcal{N}(\mathfrak{g}_{\mathbb{C}}))/K_{\mathbb{C}}$ is endowed with the closure order (??).

(ii) If $\mathcal{O}$ is a real nilpotent orbit, then $\mathcal{O}$ and $\mathcal{O}^{\vee}$ are K-equivariantly diffeomorphic.

The first property was established by Barbasch and Sepanski in [16], while the second was proved by Vergne in [17]. Each paper makes extensive use of Kronheimer's desciption of nilpotent orbits from [18].

We now prove two preliminary results, the first of which is a direct consequence of the Kostant-Sekiguchi Correspondence.

Lemma 5: If $\mathcal{O}$ is a real nilpotent orbit, then $\mathcal{O}$ is the unique G-orbit of maximal dimension in $\overline{\mathcal{O}}$.

Proof: Suppose that $\mathcal{O}'\neq\mathcal{O}$ is another G -orbit lying in $\overline{\mathcal{O}}$. By Property (i) in Theorem 4, it follows that $(\mathcal{O}')^{\vee}$ is an orbit in $\overline{(\mathcal{O}^{\vee})}$ different from $\mathcal{O}^{\vee}$. However, $\mathcal{O}^{\vee}$ is an orbit of the complex algebraic group $K_{\mathbb{C}}$ under an algebraic action, and therefore is the unique orbit of maximal dimension in its closure. Hence, $\dim_{\mathbb{R}}((\mathcal{O}')^{\vee})<\dim_{\mathbb{R}}(\mathcal{O}^{\vee})$. Property (ii) of Theorem 4 implies that the Kostant-Sekiguchi Correspondence preserves real dimensions, so that $\dim_{\mathbb{R}}(\mathcal{O}')<\dim_{\mathbb{R}}(\mathcal{O})$.

We will also require some understanding of the relationship between the G-centralizer of $\xi\in\mathfrak{g}$ and the $G_{\mathbb{C}}$-centralizer of ξ, viewed as an element of $\mathfrak{g}_{\mathbb{C}}$. Denoting these centralizers by G_{ξ} and $(G_{\mathbb{C}})_{\xi}$ respectively, we have the following lemma.

Lemma 6: If $\xi\in\mathfrak{g}$, then G_{ξ} is a real form of $(G_{\mathbb{C}})_{\xi}$.

Proof: We are claiming that the Lie algebra of $(G_{\mathbb{C}})_{\xi}$ is the complexification of the Lie algebra of G_{ξ}. The former is $(\mathfrak{g}_{\mathbb{C}})_{\xi}=\{\eta\in\mathfrak{g}_{\mathbb{C}}:[\eta,\xi]=0\}$, while the Lie algebra of G_{ξ} is $\mathfrak{g}_{\xi}=\{\eta\in\mathfrak{g}_{\mathbb{C}}:[\eta,\xi]=0\}$. If $\eta=\eta_1+i\eta_2\in\mathfrak{g}_{\mathbb{C}}$ with $\eta_1,\eta_2\in\mathfrak{g}$, then $[\eta,\xi]=[\eta_1,\xi]+i[\eta_2,\xi]$. So, $\eta\in(\mathfrak{g}_{\mathbb{C}})_{\xi}$ if and only if $\eta_1,\eta_2\in\mathfrak{g}_{\xi}$. This is equivalent to the condition that $\eta\in(\mathfrak{g}_{\xi})_{\mathbb{C}}\subseteq\mathfrak{g}_{\mathbb{C}}$, so that $(\mathfrak{g}_{\mathbb{C}})_{\xi}=(\mathfrak{g}_{\xi})_{\mathbb{C}}$.

We may now prove the main result of this section.

Theorem 5: If $\mathcal{O}_1$ and $\mathcal{O}_2$ are real nilpotent orbits with the property that $(\mathcal{O}_1)_{\mathbb{C}}=(\mathcal{O}_2)_{\mathbb{C}}$, then either $\mathcal{O}_1=\mathcal{O}_2$ or $\mathcal{O}_1$ and $\mathcal{O}_2$ are incomparable in the closure order. In other words, each fibre of $\varphi_{\mathfrak{g}}$ consists of pairwise incomparable nilpotent orbits.

Proof: Assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ are comparable. Without the loss of generality, $\mathcal{O}_1\subseteq\overline{\mathcal{O}_2}$. We will prove that $\mathcal{O}_1=\mathcal{O}_2$, which by Lemma 5 will amount to showing that the dimensions of $\mathcal{O}_1$ and $\mathcal{O}_2$ agree. To this end, choose points $\xi_1\in\mathcal{O}_1$ and $\xi_2\in\mathcal{O}_2$. Since $(\mathcal{O}_1)_{\mathbb{C}}=(\mathcal{O}_2)_{\mathbb{C}}$, we have $\dim_{\mathbb{C}}((G_{\mathbb{C}})_{\xi_1})=\dim_{\mathbb{C}}((G_{\mathbb{C}})_{\xi_2})$. Using Lemma 6, this becomes $\dim_{\mathbb{R}}(G_{\xi_1})=\dim_{\mathbb{R}}(G_{\xi_2})$. Hence, the (real) dimensions of $\mathcal{O}_1$ and $\mathcal{O}_2$ coincide.

Proof: If is surjective, and then Lemma 2 implies that is quasi-split. Also, Proposition 2 implies that each simple summand of has a surjective orbit complexification map, and the above discussion then establishes that has no simple summand of the form. Conversely, assume that is quasi-split and has no simple summand of the form. By Proposition 2 (ii), each simple summand of is quasi-split. Furthermore, the above discussion implies that the only quasi-split simple real Lie algebras with non-surjective orbit complexification maps are those of the form. Hence, each simple summand of has a surjective orbit complexification map, and Proposition 2 (i) implies that is surjective.

Acknowledgements

The author is grateful to John Scherk for discussions that prompted much of this work. The author also acknowledges Lisa Jeffrey and Steven Rayan for their considerable support. This work was partially funded by NSERC CGS and OGS awards.

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Citation: Crooks P (2016) Properties of Nilpotent Orbit Complexification. J Generalized Lie Theory Appl S2: 012. doi:10.4172/1736-4337.S2-012

This article was originally published in a special issue, **Recent Advances of Lie Theory in differential Geometry, in memory of John Nash** handled by Editor. Dr. Princy Randriambololondrantomalala, Unversity of Antananarivo, Madagascar

UNIVERSITE D’ANTANANARIVO

DOMAINE : SCIENCES ET TECHNOLOGIES

ECOLE DOCTORALE : MATHEMATIQUES ET APPLICATIONS

EQUIPE D’ACCEUIL : MATHEMATIQUES ET STRUCTURES

LABORATOIRE D’ALGEBRE ET GEOMETRIE (LAG)

LES ALGEBRES GENETIQUES ET LEURS DERIVATIONS

RANDRIAMBOLOLONDRANTOMALALA Princy

Octobre 2015

Résumé

Les mathématiques d’aujourd’hui touchent plusieurs domaines, un de leurs est la génétique. La loi de Mendel offre un ensemble muni de loi mathématiques, cet ensemble forme une algèbre appelée algèbre génétique qui est non-associative. L’étude mathématique de cette algèbre offre des interprétations considérables en génétiques. On inspecte les m-dérivations de cette algèbre qui ne sont autre qu'une généralisation de la notion de dérivations et leurs interprétations en génétique.

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UNIVERSITE D'ANTANANARIVO
Ecole Doctorale : **MATHEMATIQUES ET APPLICATIONS**
Equipe d'Accueil Doctorale : **MATHEMATIQUES ET INFORMATIQUE**
Unité de Rattachement : **LABORATOIRE D'ALGEBRE ET GEOMETRIE**

Thème : *LES DIFFERENTS DOMAINES DE RECHERCHE DANS LE LABORATOIRE D'ALGEBRE ET GEOMETRIE*

Par H. S. G. Ravelonirina et P. Randriambololondrantomalala

Directeur : Prof. ANONA Manelo Fréderic, Ambassadeur de l'ICM (International Congress of Mathematicians)-Région Océan Indien (2014-2018
Tél. +261 33 12 027 73 e-mail : mfanona@yahoo.fr
Adresse : R 212 Département de Mathématiques et Informatique, Université d'Antananarivo, BP 906, Antananarivo 101, MADAGASCAR

PRESENTATION

Le Laboratoire d'Algèbre et Géométrie ou LAG a été créé en 2002 dans le Département de Mathématiques et Informatique-Université d'Antananarivo. Ce laboratoire a pour objectif principal, non seulement de former des jeunes relèves en matière d'enseignement supérieur mais aussi capables à mener des recherches en mathématiques et leurs applications. A présent, autre que le volet principal de recherche en géométrie différentielle, ce laboratoire s'oriente petit à petit dans des divers domaines et spécialités.

PARTENAIRES
CEGMAD, FUNRECO SA, MICET.

DOMAINES DE RECHERCHE

GEOMETRIE DIFFERENTIELLE
Groupes et Algèbres de Lie, Connexions, Cohomologies, Espaces de nullité d'une courbure, m-Dérivations d'une algèbre de Lie, Algèbres génétiques, Système triple de Lie,...

GEOMETRIE DISCRETE
Polynômes d'Ehrhart, Famille des polynômes d'Higashitani, Structures algébriques et géométriques des polynômes d'Ehrhart,...

THEORIE DE DEVELOPPEMENT
Modélisation de développement, Système éducatif, Histoire de Mathématiques, Caractérologie...

UNIVERSITE D'ANTANANARIVO

FACULTE DES SCIENCES

DEPARTEMENT DE MATHEMATIQUES ET INFORMATIQUE

IZAY ADALA NO TOA AN-DRAINY

Département de Mathématiques & Informatique

DEMANDE D'ORDRE DE MISSION

NOM ET PRENOMS : RANDRIAMBOLOLONDRANTOMALALA Princy

I.M : 342 934

QUALITE ET GRADE : Professeur Titulaire

INDICE :

RESIDENCE ADMINISTRATIVE : FACULTE DES SCIENCES

LIEU DE MISSION : Fianarantsoa

OBJET DE LA MISSION : Conférence

LA MISSION EST-ELLE PRINCIPALEMENT :

- dans l'intérêt du Département ?
- dans l'intérêt du missionnaire ?
- dans l'intérêt des deux ?

DATE DE DEPART : 16 juin 2015

DATE DE RETOUR : 19 juin 2015

MOYEN DE TRANSPORT :

IMPUTATION BUDGETAIRE (Indemnité de déplacement) :

Budget :………………………………….. Article :………………………………….

Chapitre :………………………………… Paragraphe :……………………………..

Vu au passage à Fianarantsoa

LE DOYEN

Randriamirina Benjamin

Antananarivo, le 12 juin 2015…

L'intéressé(e),

Avis du Chef de Département

Armand RAMAMONJISOA

Avis du Doyen

Le Doyen de la Faculté des Sciences,

RAHERIMANDIMBY Marson

Professeur Titulaire

Département de Mathématiques et Informatique
Faculté des Sciences – Université d'Antananarivo
BP 906
101 Antananarivo
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UNIVERSITE D'ANTANANARIVO

DOMAINE : SCIENCES ET TECHNOLOGIES

ECOLE DOCTORALE : MATHEMATIQUES ET APPLICATIONS

EQUIPE D'ACCEUIL : MATHEMATIQUES ET STRUCTURES

LABORATOIRE D'ALGEBRE ET GEOMETRIE (LAG)

LES CHAMPS DE VECTEURS POLYNOMIAUX SUR LES VARIETES ANALYTIQUES

RANDRIAMBOLOLONDRANTOMALALA Princy

Juin 2015

Résumé

On inspecte les algèbres de Lie de champs de vecteurs polynomiaux contenant localement tous les champs constants et le champ d'Euler dans une variété analytique réelle et puis dans une variété holomorphe. Plus précisément, on choisit quelques variétés analytiques réelles et analytiques complexes compactes assez connues comme S^n ou T^n… avec n un entier plus grand ou égal à un. Ces études sont les continuations de celles de [3,4,5].

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UNIVERSITE D'ANTANANARIVO
ECOLE DOCTORALE : MATHEMATIQUES ET APPLICATIONS
DEPARTEMENT DE MATHEMATIQUES ET INFORMATIQUE

LABORATOIRE D'ALGEBRE ET GEOMETRIE

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DOMAINES DE RECHERCHE

1° GEOMETRIE DIFFERENTIELLE : Groupes et Algèbres de Lie, Connexions, Espaces de nullité d'une courbure, m-Dérivations d'une algèbre de Lie, Algèbres de Lie à la Biomathématique, Système triple de Lie,...
2° GEOMETRIE DISCRETE : Polynômes d'Ehrhart, Famille des polynômes d'Higashitani, Structures algèbriques et géométriques des polynômes d'Higashitani,...
3° THEORIE DE DEVELOPPEMENT : Modélisation économique, Système éducatif, Histoire de Mathématiques,...

THESES EN COURS

- Sur la famille de polynômes d'Higashitani et Contre-exemples de la Conjecture de Beck et autres. (Rakoto Jean Jolly)
- Sur les Algèbres de Lie définies par une 1-forme vectorielle. (Andrianantenaina Roddy S.)
- Sur les Connexions méromorphes. (Randriatsarafara Hobiniaina)
- Sur les Connexions provenant d'une métrique (Lantonirina Sendrasoa Laurence)
- Sur les Variétés finslériennes à courbure régulière de dimension 3. (Fahad Ben Abdou)
- Sur les Connexions correspondantes à une courbure régulière homogène. (Ratovoarimanana Hasina)

PARTENAIRES
CEGMAD, FUNRECO SA, MICET.

ECOLE DE COMBINATOIRE ET ALGORITHMIQUE DE MADAGASCAR
ECAM 2012

3-14 septembre 2012
UNIVERSITE D'ANTANANARIVO

Membres du Comité scientifique international :

Fanja Rakotondrajao, Maitre de Conférences, Université d'Antananarivo, Co Chair
Roberto Mantaci, Maitre de Conférences, Université Paris 7, France, Co Chair
Jean-Baptiste Yunes, Maitre de Conférences, Université Paris7, France
Mike Zabrocki, Assistant Professeur, Université de York, Canada
Stephan Wagner, Assistant Professeur, Université de Stellenbosch, Afrique du Sud

Membres du Comité d'organisation :

Fanja Rakotondrajao, Université d'Antananarivo
André Totohasina, Université d'Antsiranana
Benjamin Randrianirina, Université de Fianarantsoa
Falimanana Randimbindrainibe, ESPA, Université d'Antananarivo
Elysé Rajaonarimanana, ENS Antananarivo
Princy Randriambololondrantomalala, Université d'Antananarivo

Présentation de l'ECAM

Cette Ecole a été initiée dans le cadre du développement de recherche en Mathématiques plus particulièrement en Combinatoire à Madagascar, en brisant notre isolement et en favorisant les contacts de nos étudiants et chercheurs avec les experts internationaux, ainsi qu'en favorisant les échanges des chercheurs dans la région. La première Édition tenue à l'Ecole Supérieure Polytechnique d'Antananarivo en 2011 a été très fructueuse. Les participants ont manifesté leur souhait pour la pérennisation de telle manifestation.

Thème

Combinatoire analytique appliquée à l'analyse réelle et complexe de l'étude des objets combinatoires.

Contacts:

Pour plus d'informations, veuillez contacter l'organisateur :
email : ***emamada@gmail.com*** ***ou*** ***frakoton@yahoo.fr***
tel : + 261320419582

Public cible

Cette Ecole s'adresse aux étudiants titulaires de maitrise ou master (DEA), doctorants et enseignants-chercheurs des départements de mathématiques et d'informatique ainsi que aux ingénieurs en Telecom des Universités d'Antananarivo, d'Antsiranana et de Fianarantsoa ainsi que des pays de la zone Afrique (Est, Austral et Océan Indien).

Frais de participation

Le frais de participation à l'atelier est 20.000 Ar (vingt mille Ariary). Des bourses, en nombre limité, pourraient être offertes sur une base sélective pour couvrir les frais d'inscription, de transport et de séjour de certains participants.

Date de dépôt de candidature (CV et lettre de motivation)

Avant le ***20 Aout 2012***

Les candidats sélectionnés seront avisés le vendredi 31 aout 2012 pour l'inscription définitive avec le paiement des frais le 3 septembre 2012.

CERTIFICAT

Randriambololondrantomalala

a participé à l'

Ecole Mathématique en Afrique

Première édition

tenue à l'Ecole Supérieure Polytechnique

de l'Université d'Antananarivo, Madagascar

du 19 au 30 Septembre 2011

Thèmes : ***Théorie des Nombres et Cryptologie,***
Algorithmique,
Modélisations

Les Co Chairs du Comité d'Organisation

Professeur Saliou Touré
Président de l'Union Mathématique Africaine

Docteur Fanja Rakotondrajao
Présidente de la Société Mathématique Malagasy

LIRIMA - UMA - SMMG - STIC-Mada

Centre international de rencontres mathématiques

163 avenue de Luminy, Case 916
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Rencontres 2012

Les couleurs indiquent le type de rencontre

Colloques et écoles **Petits groupes** **Recherche en binôme**
Session thématique

- du 18 juin au 22 juin 2012

  Les systèmes d'Anosov

  Organisateurs :

  - Thierry BARBOT (Univ. Avignon)
  - Sergio FENLEY (Florida State, Princeton University)
  - Boris HASSELBLATT (Tufts University)
  - Carlos Alberto MAQUERA APAZA (University of Sao Paulo)
  - Abdelghani ZEGHIB (ENS-Lyon)

Liste des participants à la rencontre

- **ASAOKA Masayuki**, Kyoto University

- **BARBOT Thierry**, Univ. Avignon
- **BARTHELME Thomas**, Penn State University
- **BEGUIN François**, Univ. Paris-Sud
- **BELRAOUTI Mehdi**, Univ. Avignon
- **BENOIST Yves**, Univ. Paris-Sud
- **BONATTI Christian**, Univ. Bourgogne
- **BONTHONNEAU Yannick**, ENS Paris
- **BRAVO VIDARTE José Humberto**, University of Sao Paulo
- **CRAMPON Mickael**, La Chambotte
- **DEHORNOY Pierre**, Univ. Grenoble I
- **FENLEY Sergio**, Florida State, Princeton University
- **FOULON Patrick**, CIRM
- **GUERITAUD François**, Univ. Lille 1
- **HASSELBLATT Boris**, Tufts University
- **KANAI Masahiko**, University of Tokyo
- **KASSEL Fanny**, Univ. Lille 1
- **MAQUERA APAZA Carlos Alberto**, University of Sao Paulo
- **MONCLAIR Daniel**, ENS Lyon
- **MORENO Matthias**, ENS - Lyon
- **PATERNAIN Miguel**, University of Republica - Urugay
- **PINSKY Tali**, Technion University
- **PONCE Gabriel**, University of Sao Paulo
- **RANDRIAMBOLOLONDRANTOMALALA Princy**, ENS - Lyon
- **RAVELONIRINA H. Sammy Grégoire**, ENS - Lyon
- **RECHTMAN Ana**, Univ. Strasbourg
- **RODRIGUEZ HERTZ Federico**, The Pennsylvania State University
- **RODRIGUEZ HERTZ Jana**, University of Republica - Urugay
- **ROSSI-SALVEMINI Clara**, Univ. Avignon
- **SIMIC Slobodan**, San Jose State University
- **TINTAYA MOLLO Renato Alejandro**, University of Sao Paulo
- **TROUBETZKOY Serge**, Aix-Marseille Univ.
- **VALDIVIA Aaron**, Florida State University
- **VARAO FILHO Jose Regis**, University of Sao Paulo
- **VERJOVSKY Alberto**, University of Cuernavaca
- **WALLER Russell**, Florida State University
- **YU Bin**, Univ. Bourgogne
- **ZEGHIB Abdelghani**, ENS Lyon

38 participants
Mis à jour le 04/11/2014.

Proposal – Simons Foundation Africa Project:
Differential Geometry and Combinatorics
Antananarivo, Madagascar

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(same address as Academic co-Directors)

International contacts (research and/or training):

Dr. Abdelghani Zeghib (ENS Lyon, France)
Dr. Mike Zabrocki (York University, Toronto, Canada)
Dr. Jean-Baptiste Yunès (LIAFA, Paris 7, France)
Dr. Stephan Wagner (Stellenbosch University, South Africa)
Dr. Konrad Polthier (Freie Universität Berlin, Germany)
Dr. Roberto Mantaci (LIAFA, Paris 7, France)
Dr. Ingrid Daubechies (Duke University, USA)

TABLE OF CONTENTS

SIMONS FOUNDATION

The Simons Foundation Africa Mathematics Project (AMP)
Instructions to Finalists

The Simons Foundation congratulates the participants in the project outlined in your Concept Paper. Your project has been selected as one of the 12 finalists in the competition for Foundation funding under its Africa Mathematics Project. From among the 12 finalists, AMP will award approximately five grants, each 80,000 USD per year for five years, to research groups in mathematics at five universities in diverse regions across sub-Saharan Africa. A similar second round of five-year grants, comprised of new awards and/or award renewals, will be considered for 2017.

AMP therefore invites you to submit a final proposal as the second and final step of the two-step grant application process. Deadline for submission of final proposals is **June 1, 2012**. The AMP Scientific Committee will once again, review these proposals. Announcement of awards is planned on or about September 1, 2012.

Consultations: Professors Herb Clemens and Wandera Ogana of the African Mathematics Project Planning Committee have been asked to serve as your contacts during your preparation of your final proposal. You are encouraged to contact them with any questions you might have. They are not members of the AMP Scientific Committee and will not be privy to its deliberations. They are; therefore, free to give you guidance and advice without compromising objectivity of the final selection process.

- Herb Clemens; clemens@math.ohio-state.edu
- Wandera Ogana; wogana@uonbi.ac.ke

Final proposal contents and format: 15-page (maximum) concept narrative (single or double-spaced, 12-point font). The cover page should include the name of the proposed research project, as well as the Lead Researcher's name, affiliation, mailing address, telephone and email address. The Lead Researcher shall serve as Academic Director of the project. If the project contact is other than the Academic Director, such as the Project Secretariat, include his or her name, mailing address, telephone and email address as well.

Page 2 should state the overall research goal of the project, followed by a succinct and focused statement describing the research group's two or three main objectives and a short description of its operation.

Pages 3–15 should be a focused and crisp narrative containing the elements listed under "Required proposal elements" and "Strongly suggested proposal elements" listed below. This narrative should be more focused and detailed than the concept paper. If more than one research or training area is listed, the different areas should be prioritized and partners responsible for each area and aspect should be identified. *Research areas in particular should be limited to those that directly relate to the overall research goal of the project. Inclusion of several areas, especially areas only tenuously related to each other will lessen the chances that a proposal will be funded.*

In addition to the 15-page concept narrative, the following additional items should be submitted:

1. A five-page budget, one page for each 6 months of the first 2.5 years of the grant, itemized for each of the 8 eligible categories of funding listed below. The final total of expenditures for the first two and a half years of operation should not exceed $200,000. *Salary supplements should not exceed 40% of the annual grant total during any year of the grant and should be justified on the basis of an explicitly stated percentage of released time from other*

professional duties.

2. An additional one-page explanation of budget categories should also be submitted. This page should indicate additional sources of financial or in-kind support for the project.
3. Curriculum vitae of researchers to be funded under the proposal. Individual CV's should be no longer than 4 pages and should focus on education, positions, activities and publications relevant to the proposed research.
4. List of academic institutions and principal collaborators participating in grant activities, together with any current or pending research support from other sources applicable to the research proposed.
5. Official letter of support from the vice-chancellor of the applicant research group's university.

Required proposal elements:

1. Identification of a *single* mathematical area that is to be the principal focus of proposed research and postgraduate training. Explanation of how any proposed research and training in other (related) areas serves the main research and training goals of the project. If more than one research group or mathematical area is listed, care must be taken to fully justify a close relationship with the common focus of the proposed research. The inclusion of researchers, however distinguished, whose work is not directly supportive of, and closely related to the central research focus, is discouraged.
2. Individual, group and institutional capacity for carrying out the proposed research and training. (Group may include several universities in the region.)
3. Detailed research and training plan to be carried out over a five-year period together with its rationale. Identification of the individual(s) primarily responsible for each aspect of the work.
4. Program outline and projected timeline for studies of participating graduate students.
5. Estimated outcomes:
    a. research deliverables (e.g. approximate number of publications in

international journals expected)

b. training deliverables (e.g. approximate number of PhD and of master's degrees expected)

6. A networking plan for contact with other centers or projects on the continent that have compatible mathematical interests and objectives.
7. A plan for a regional workshop in the project's research area around the end of the second year.
8. Commitment of in-kind matching support from the applicant group's university (e.g., classroom or office space, reliable internet connectivity, conference hosting and support).
9. Detailed plan for an efficient and transparent system for budget administration, disbursement and reporting.

Strongly suggested proposal elements:

1. Identification of international research partner(s), including those from South Africa and Mediterranean African countries.
2. Project participation in an existing mathematical research network with a compatible mathematical focus.
3. Plan for sustainability of the project through an additional five-year period.
4. Strategy for retaining at universities in the region both faculty members whose research is supported under this grant and postgraduate students whose advanced degree training is supported under this grant.
5. Plan for incorporating women mathematicians in the project and/or identifying and nurturing female mathematical talent.

Finally, proposers are invited to consider the preparation of a 5-10-minute video or other visual presentation introducing themselves and their proposal to the Scientific Committee and, if feasible, the physical locations where the research activities will be located.

Categories eligible for funding:

1. Graduate training (tuition, stipends, housing, etc.)

2. Salary supplements for faculty for research and for mentoring of Ph.D. and master's degree students (total in this item is not to exceed 40% of annual grant amount)
3. Travel and conferences
4. Stipends and expenses for visiting faculty and students
5. Equipment (including maintenance and servicing) and supplies
6. Books, journals, databases, etc.
7. Communications
8. Secretariat office expenses (not to exceed 10% of annual grant amount)

Duties of project personnel:

Academic Director: Each project will designate an Academic Director, based at one of the participating universities, to provide overall guidance for, and coordination of, the project's training and research activities. The Academic Director must be a leader and motivator, a practicing scientist with stature in the academic community and strong management skills. He or she will serve as Principal Investigator for the AMP proposal and will be responsible for preparing annual reports and assessments of the network's operations.

Secretariat: Each project will include a Secretariat responsible for all administrative details, including travel for faculty, students and visitors, workshop/conference planning, and financial management of the project. The Secretariat may be located in the same academic unit as the Academic Director, but this is not a requirement. Projects are encouraged to adopt whatever structure will best ensure that both academic and administrative leadership are effective

Review criteria: Selection will be conducted through a peer-reviewed application process. The primary selection criteria will be scientific merit, training capacity and the quality of research activities. Evidence of institutional support, networking, and potential for sustainability will also be very important.

Reporting Requirements: A yearly progress report will be required, as well as the submission of the second half of the detailed budget after the first two years.

Continued support after the first two and a half years will be contingent on the ongoing evaluation of the project.

Final proposal submission: Proposals should be submitted electronically, as email attachments in PDF format, to:

The Simons Foundation Africa Mathematics Project
c/o Ms. Meghan Criswell
E-mail address: mcriswell@simonsfoundation.org

Submission deadline: June 1, 2012

Contrat de Collaboration

Entre

Ingrid Daubechies

Et

Institute for the Conservation of Tropical Environments à Madagascar, ci-après désignée « ICTE », ayant sa Représentation Nationale à Manakambahiny, Villa ROTCIV, Lot VU 283 D -ANTANANARIVO 101, représenté par Dr. Benjamin ANDRIAMIHAJA, agissant en qualité et en vertu des pouvoirs qui lui sont conférés,

Objet de la collaboration

Cette collaboration entre dans le cadre de la Formation des étudiants de troisième cycle et du support de la recherche effectuée par des chercheurs de la filière Mathématiques, Faculté des Sciences de l'Université d'Antananarivo.

Modalités de mise en œuvre

Les étudiants et chercheurs qui bénéficieront des bourses seront choisis par les professeurs Anona et Rakotondrajao du Département de Mathématique de l'Université d'Antananarivo et Pr. Ingrid Daubechies ; ce choix (ainsi que le montant des bourses) sera ré-évalué tous les six mois. Ce montant ne devrait pas excéder $150/mois pour chaque bourse. Une bourse attribuée à un(e) étudiant(e) peut etre interrompue à tout instant si l'effort de recherche est insuffisant. (Voir appendice pour les critères d'application.)

La liste des étudiants et chercheurs bénéficiaires d'une bourse sera envoyé au bureau de MICET

Les étudiants pourront utiliser la connection internet du bureau de l'ICTE à Manakambahiny, et réserver un local pour discussions hebdomadaires.

Les bourses commenceront au plus tot le 1er décembre 2012, mais peuvent débuter plus tard.

Obligations des parties contractantes

Ingrid Daubechies : assurera la recherche des fonds

ICTE :

ICTE assurera la distribution des bourses aux étudiants après approbation des Pr Anona, Rakotondrajao et Daubechies

ICTE effectuera un rapport financier qui sera envoyé au bailleur tous les 6 mois.

Les frais de gestion sont calculés sur une base de « overhead » de 15% (c.à.d. que le bailleur attribue $15 à ICTE pour chaque $100 déboursé aux étudiants et chercheurs), ce qui revient à 15/115 du montant total des fonds, ou environ 13.04% .

Durée de la collaboration

Le présent Accord de collaboration est effectif à partir de la disponibilité jusqu'à l'épuisement des fonds.

Règlement des conflits

En cas de différends dans l'interprétation et l'exécution de la présente convention, les parties s'engagent à les régler à l'amiable.

octobre

Fait à Antananarivo, le 5 ~~mai~~ 2012

Lu et approuvé par :

Pr ANONA Mahelo Frédéric

RAKOTONDRAJAO Faniya

Ingrid DAUBECHIES.

Lu et approuvé par :

INSTITUTE FOR THE CONSERVATION OF TROPICAL ENVIRONMENTS ICTE

ANDRIAMIHAJA Benjamin

ÉCOLE NORMALE SUPÉRIEURE DE LYON

15 parvis René-Descartes
BP 7000, 69342 Lyon cedex 07
Tel. +33 (0)4 37 37 60 00
Fax +33 (0)4 37 37 60 60
www.ens-lyon.fr

cnrs
dépasser les frontières

Unité de Mathématiques Pures et Appliquée
Site Monod - 46 Allée d'Italie – 69364 Lyon Cedex 07
☎ : 04 72 72 84 24/84 85 - Fax : 04 72 72 84 80
: secretariat@umpa.ens-lyon.fr

Lyon, le 14 mars 2012

ATTESTATION

Je soussigné, Laurent BERGER, Directeur de l'Unité de Mathématiques Pures et Appliquées de l'ENS-Lyon, certifie que Monsieur Princy Randriambololondrantomalala est invité par notre laboratoire dans le cadre d'une collaboration scientifique du 15 avril 2012 au 30 juin 2012, séjour pris en charge intégralement par l'Université de Lyon. Pendant tout le séjour Mr Randriambololondrantomalala sera domicilié à l'ENS :

ENS-Lyon Site Monod

Résidence Etudiante

46 allée d'Italie

69364 Lyon Cedex 07

Fait à Lyon, le 14 mars 2012,

Pour servir et valoir ce que de droit

Le Directeur de l'UMPA,

Laurent BERGER

Université de Liège

Faculté des Sciences
Département de Mathématique

Géométrie et théorie des algorithmes
Professeur Pierre Lecomte

Liège, le 1er octobre 2014

Monsieur P. Randriambololondrantomalala
Département de Mathématique et Informatique
Faculté des Sciences
Université Antananarivo
P.B. 906 Ankasko, Madagascar

Monsieur,

Suite à votre message du 25 septembre dernier, et sous réserve que votre demande de financement soit honorée par l'Administration adhoc, j'aurais le plaisir de vous accueillir dans mon équipe pour un séjour d'un mois dans le cadre staff mobility du programme Erasmus-Mundus dream.up.pt afin de travailler avec vous sur les sujets de recherches que vous m'avez mentionnés.

Idéalement, ce séjour pourrait prendre place au premier semestre d'une année académique, soit entre le 15 septembre et le 15 décembre, et par exemple en 2015, car ma charge de cours est bien plus légère dans cette période de l'année que durant le second semestre.

En espérant que vous pourrez venir,

Cordialement,

Pierre Lecomte
Professeur

12 Grande Traverse, Bât. B37, parking 32
Tél. +(32)-4- 366 93 83 Fax. +(32)-4- 366 95 47 E-mail : plecomte@ulg.ac.be

UNIVERSITE D'ANTANANARIVO
FACULTE DES SCIENCES
DEPARTEMENT DE MATHEMATIQUES ET INFORMATIQUES

Mémoire en vue de l'obtention du
DIPLOME D'ETUDES APPROFONDIES
Option : Mathématiques pures
Spécialité : Géométrie différentielle

CONNEXIONS ET ESPACE DE NULLITE DE LA COURBURE

Présenté par : RANDRIATSARAFARA Iharintsoa Hobiniaina

Soutenu le 16 Décembre 2014

MEMBRES DU JURY

Président :	Fréderic Manelo ANONA Professeur Titulaire à l'Université d'Antananarivo
Encadreur :	Princy RANDRIAMBOLOLONDRANTOMALALA Maître de Conférences à l'Université d'Antananarivo
Examinateurs :	- Hanitriniaina Sammy Grégoire RAVELONIRINA Maître de Conférences à l'Université d'Antananarivo
	- Joseph RAKOTONDRALAMBO Maître de Conférences à l'Université d'Antananarivo

UNIVERSITÉ D'ANTANANARIVO

Faculté des sciences

Département de Mathématiques et Informatique

Mémoire en vue de l'obtention du

Diplôme d'Etudes Approfondies

Spécialité : Mathématiques Appliquées

Option : Géométrie Différentielle

Sur les variétés riemanniennes

Présenté par : Sendrasoa Laurence LANTONIRINA

Soutenu le 30 Avril 2015

MEMBRES DE JURY

Président :	- Fréderic Mandelo ANONA Professeur Titulaire à l'Université d'Antananarivo
Encadreur :	- Princy RANDRIAMBOLOLONDRANTOMALALA Maître de Conférences à l'Université d'Antananarivo
Examinateur :	- Hanitriniaina Sammy Grégoire RAVELONIRINA Maître de Conférences à l'Université d'Antananarivo

UNIVERSITE D'ANTANANARIVO
FACULTE DES SCIENCES
DEPARTEMENT DE MATHEMATIQUES ET INFORMATIQUE

Mémoire en vue de l'obtention du

Diplôme d'Etudes Approfondies

Spécialité : Mathématiques Appliquées

Option : Géométrie Différentielle

Les formes vectorielles et connexions de Grifone appliquées aux variétés finsleriennes à courbure régulière

Présenté par

Fahad Ben ABDOU

Soutenu le 30 Avril 2015

Président de jury :	Monsieur Fréderic Manelo ANONA, Professeur Titulaire à l'Université d'Antananarivo
Encadreur :	Monsieur Princy RANDRIAMBOLOLONDRANTOMALALA, Maître de Conférences à l'Université d'Antananarivo
Examinateur :	Monsieur Hanitriniaina Sammy Grégoire RAVELONIRINA, Maître de Conférences à l'Université d'Antananarivo

UNIVERSITE D'ANTANANARIVO

FACULTÉ DES SCIENCES

Département : Mathématiques et Informatique

Mention : Mathématiques et Applications

Parcours : Mathématiques et Structures

Laboratoire : Algèbre et Géométrie

Spécialité : Géométrie Différentielle

Memoire en vue de l'obtention du **Diplôme de Master II**

SUR LES VARIÉTÉS ANALYTIQUES

Présenté par : **RANDRIAMIDOSY Harimanana Roger**, le 11 février 2016, devant la commission d'examen formée de

Président de Jury	-	Monsieur Frédéric Maneto ANONA Professeur Titulaire à l'Université d'Antananarivo
Rapporteur	-	Monsieur Princy RANDRIAMBOLOLONDRANTOMALALA Maître de Conférences à l'Université d'Antananarivo
Examinateurs	-	Monsieur Hanitriniaina Sammy Gregoire RAVELONIRINA Maître de Conférences à l'Université d'Antananarivo
	-	Monsieur Armand RAMAMONJISOA Maître de Conférences à l'Université d'Antananarivo.

Code : **SMIM12H0CS8** Intitulé de l'UE **: Homologie et cohomologie** Type et statut de l' UE: Optionnel Responsable de l'UE **: Randriambololondrantomalala Princy**, Maitre de conférences d'ESR , spécialité : Géométrie différentielle, Contacts : 0331246249 princypcpc@yahoo.fr	
Charge totale du travail étudiant 150	
***Nombre de crédits* : 5**	
Semestre d'études : 8	
Compétences visées et évaluation des compétences: Voir les Outils modernes de géométrie, d'analyse et d'algèbres à travers l'homologie et cohomologie.	
Contenu et objectif de l'UE : Familiariser avec les groupes d'homologie et de cohomologie, voir les applications dans les mathématiques modernes.	
Nombre minimal d' étudiants pour autoriser l' ouverture de l' UE : 5	
Conditions d'attribution des crédits : La participation régulière et active à tous les cours de l'UE ainsi que la réussite aux contrôles de connaissances.	
Evaluation de l'UE : Examen final écrit Une note inférieure ou égale à 05/20 (note éliminatoire) à une ECUE entraîne l'absence de note à l'UE	
Intitulé des éléments constitutifs/matières constituant l'UE : 1 - Rappels généraux et introduction sur l'homologie et cohomologie 2 – Quelques groupes d'homologies et de cohomologie	
1	**Intitulé de l'EC :** Rappels généraux et introduction sur l'homologie et cohomologie
a	**Responsable de l'EC : Ravelonirina H. S. G.**, Maître de Conférences, spécialité : Géométrie différentielle Contacts : 0331121675 rhsammy@yahoo.fr
b	**Taille de groupe :** Cours : 300 étudiants TD : 25 par groupe

c	**Pré-requis :** Algèbre associative ou non associative, un aperçu de la théorie des catégories.
d	**Forme d'examen:** Epreuve écrite sur table.
e	**Compétences visées et évaluation des compétences** : Les étudiants doivent savoir le but de l'étude de l'homologie et de la cohomologie en vue de ses applications.
f	**Contrôle des connaissances :** ET : Examen terminal
g	**Fréquence de l'offre :** Chaque semestre
h	**Nature des activités pédagogiques en présentiel :** C, TD.
i	**Contenu du cours et documentation pour l'étudiant :** 1) Définitions et problèmes fondamentaux 2) Les espaces d'homologie et de cohomologie en générales **Documentation** 1° C.Weibel, An introduction to homological algebra. Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, Cambridge 1994. 2° Thierry MASSON, Introduction à l'homologie et la cohomologie avec exemples (webographie) 3° Bourbaki : "Topologie Générale" 4° Frédéric Paulin : "Topologie algébrique élémentaire"FIMFA (Webographie) 5° H. Cartan et S. Eilenberg: Homological Algebra" 6° A. Wallace : "Algebraic Topology : homology and cohomology" 7° André Gramain : "Topologie des surfaces"
j	**Description du travail personnel de l'étudiant :** Préparation et approfondissement du cours, travail individuel et/ou en équipe, recherche de ressources documentaires/ d'informations (dans une bibliothèque, par internet ···), établissement de fiches de lecture.
k	**Eléments de pédagogie** *(par EC)* : I. Démarche déductive II. Méthode : démonstration III. Techniques pédagogiques : démonstration, enseignement magistral et dirigé IV. Cours en ligne Outils : cours polycopiés, cours projetés, documentation (ouvrage et internet).

2	**Intitulé de l'EC :** Quelques groupes d'homologies et de cohomologie
a	**Responsable de l'EC : Randriambololondrantomalala Princy**, Maitre de conférences d'ESR , spécialité : Géométrie différentielle, Contacts : 0331246249 princypcpc@yahoo.fr
b	**Taille de groupe :** Cours : 300 étudiants TD : 25 par groupe
c	**Pré-requis :** Un peu d'Algèbres de Lie, Géométrie différentielles, topologie générales
d	**Forme d'examen:** Epreuve écrite sur table.
e	**Compétences visées et évaluation des compétences** : Voir les applications du groupe d' homologie ou de cohomologie de de Rham et Chevalley-Eilenberg et autres.
f	**Contrôle des connaissances :** ET : Examen terminal
g	**Fréquence de l'offre :** Chaque semestre
h	**Nature des activités pédagogiques en présentiel :** C, TD.
i	**Contenu du cours et documentation pour l'étudiant :** 1) Groupe d'homologie et de cohomologie de de Rham et ses applications 2) Groupe de cohomologie de Chevalley-Eilenberg et ses applications 3) Autres groupes d'homologie et de cohomologie , ses applications. **Documentation** 1° C.Weibel, An introduction to homological algebra. Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, Cambridge 1994. 2° Thierry MASSON, Introduction à l'homologie et la cohomologie avec exemples (webographie) 3° Bourbaki : "Topologie Générale" 4° Frédéric Paulin : "Topologie algébrique élémentaire"FIMFA (Webographie) 5° H. Cartan et S. Eilenberg: Homological Algebra" 6° A. Wallace : "Algebraic Topology : homology and cohomology" 7° André Gramain : "Topologie des surfaces"

j	**Description du travail personnel de l'étudiant :** Préparation et approfondissement du cours, travail individuel et/ou en équipe, recherche de ressources documentaires/ d'informations (dans une bibliothèque, par internet ⋯), établissement de fiches de lecture.
k	**Eléments de pédagogie** *(par EC)* **:** I. Démarche déductive II. Méthode : démonstration III. Techniques pédagogiques : démonstration, enseignement magistral et dirigé IV. Cours en ligne Outils : cours polycopiés, cours projetés, documentation (ouvrage et internet).

Intitulé de la MASTER: Mathématiques et Informatique

Intitulé de l'UE : **Analyse mathématique 1**

Type et statut de l'U.E*. : Mathématiques appliquées*

Code et numéro de l'UE **SMIM11AMAS7**	Charge de travail étudiant 180	Nombre de crédits 6	Semestre d'études 7	Durée *1*

<table>
<tr>
<td>1</td>
<td>Intitulé des éléments constitutifs
A). Compléments de topologie
B). Calcul variationnel, forme différentielle
C). Variétés et algèbre de Lie</td>
<td>En présentiel
VH hebdo.=4,28
VH semestriel=60</td>
<td>Travail personnel de l'étudiant
Exprimé en heures semestrielles
120</td>
<td>Coefficient
5</td>
<td>Crédits
Compter pour un crédit entre 20 à 25 heures de travail de l'étudiant, jumelant le travail présentiel, le travail personnel et les examens</td>
</tr>
<tr>
<td>2</td>
<td colspan="5">Formes d'enseignement : Cours, TD</td>
</tr>
<tr>
<td>3</td>
<td colspan="5">Taille de groupe : 25 étudiants par groupe</td>
</tr>
<tr>
<td>4</td>
<td colspan="5">Compétences visées et évaluation des compétences :</td>
</tr>
<tr>
<td>5</td>
<td colspan="5">Contenu et objectif de l'UE donner des thèmes mathématiques adaptés aux futurs probabilistes et staticiens.</td>
</tr>
<tr>
<td>6</td>
<td colspan="5">Documentation
- N. Bourbaki, Topologie générale, chap 1-4, Masson, 1990
- N. Bourbaki, Topologie générale, chap 5-8, Herman, 1974
- N. Bourbaki, Espaces vectoriels topologiques, chap 1-5, Masson, 1981
- F. Paulin, Topologie, analyse et calcul différentiel, MAF 2008-2009
- H. Cartan, Formes différentielles, Applications élémentaires au calcul des variations et à la théorie des courbes et des surfaces, Herman, Paris 1967
- J. Dugundji, Topology, Allyn and Bacon Inc, Boston, Mars 1966
- Y. Choquet-Bruhat, Géométrie différentielle et système extérieurs, Dunod, 1968
- S. Lang, Introduction aux variétés différentiables
- N. Bourbaki, groupes et algèbre de Lie, Herman
- P. Malliavin, géométrie différentielle intrinsèque</td>
</tr>
<tr>
<td>7</td>
<td colspan="5">Eléments de pédagogie: Démarche déductive et méthode participative.</td>
</tr>
<tr>
<td>8</td>
<td colspan="5">Pré-requis : Licence de mathématiques</td>
</tr>
</table>

9	**Formes d'examen** : *Epreuve écrite sur table* **Contrôle des connaissances** : *examen final* **Notes de l'UE** : *(Moyenne des notes des différents contrôles de connaissances, pondérée selon le nombre de crédits des cours, note éliminatoire)*
10	**Conditions d'attribution des crédits** : Participation régulière et active à tous les cours de l'UE ainsi que réussite aux contrôles de connaissances.
11	**Fréquence de l'offre** : chaque semestre / tous les deux semestres
12	**Responsable de l'UE et enseignant responsable des éléments constitutifs : RANDRIAMBOLOLONDRANTOMALALA Princy**
13	**Autres informations** : *(éventuellement)*

Intitulé de la MASTER: Mathématiques et Informatique

Intitulé de l'UE: **Variétés riemanniennes**

Type et statut de l'U.E. *fondamental*

Code et numéro de l'UE SMIM11VRS8	Charge de travail étudiant 150	Nombre de crédits 5	Semestre d'études 8	Durée *1*

1	**Intitulé des éléments constitutifs** A) Métriques riemannienne sur une variété B) Connexions sur une variété	En présentiel *VH hebdo.=3,57* *VH semestriel=50*	Travail personnel de l'étudiant *Exprimé en heures semestrielles* 100	Coefficient 5	Crédits *Compter pour un crédit entre 20 à 25 heures de travail de l'étudiant, jumelant le travail présentiel, le travail personnel et les examens*
2	**Formes d'enseignement** : Cours, TD, TPE				
3	**Taille de groupe** : 25 étudiants par groupe				

4	**Compétences visées et évaluation des compétences** : Avoir une base profonde sur la théorie et applications de la géométrie riemannienne
5	**objectif de l'UE et contenu** de **l'EC :** Montrer la généralisation de la géométrie euclidienne par celle de Riemann, ses belles applications en mathématiques, en mécaniques et physique théorique.
6	**Documentation** **-** Chris Peters : Geométrie differentielle I et Geometrie Differentielle II (Webographie) - G. de Rham : Variétés différentiables - F. Paulin " Géométrie Differentielle élémentaires" (Webographie) - Gilles Carron : " cours de Géométrie différentielles de M1 : notes de cours et exercices" (webographie) - Auslander et MacKenzie : " Introduction to Differentiable Manifolds" - F.W Warner : "Foundantions of Differentiable manifolds and Lie groups" - S-S Chern, Z Shen, Riemann-Finsler Geometry - F. Paulin, Leçons de géométrie riemannienne,
7	**Eléments de pédagogie :** D). Démarche déductive. E). Méthode : démonstration, magistrale. F). Techniques pédagogiques : démonstration, enseignement magistral. G). Outils : cours polycopiés, cours projetés, documentation (ouvrage et internet).
8	**Pré-requis :** Variétés différentielles, Calculs différentiels sur les variétés.
9	**Formes d'examen** : *Epreuve écrite sur table* **Contrôle des connaissances** : *examen final* **Notes de l'UE** : *Moyenne des notes des différents contrôles de connaissances, pondérée selon le nombre de crédits des cours, note éliminatoire*
10	**Conditions d'attribution des crédits** : Participation régulière et active à tous les cours de l'UE ainsi que réussite aux contrôles de connaissances.
11	**Fréquence de l'offre** : chaque semestre
12	**Responsable de l'UE et enseignant responsable des éléments constitutifs : RANDRIAMBIOLOLONDRANTOMALALA Princy,** Géométrie différentielle, +261331246249, princypcpc@yahoo.fr
13	**Autres informations** : *(éventuellement)*

Code et intitulé UE : Géométrie et Equations différentielles. Type et statut de l'UE: Obligatoire Responsable de l'UE : **Randriambololondrantomalala Princy**, Maître de Conférences d'ESR en activité, Géométrie différentielle, princypcpc@yahoo.fr , **Tél**.:+261331246249
Charge totale du travail étudiant : 90h
Nombre de crédits **: 3**
Semestre d'études : 2
Compétences visées et évaluation des compétences : Familiariser les étudiants avec l'approche géométrique dans la résolution des problèmes mathématiques.
Objectif de l'UE : ***Donner*** aux étudiants les outils de base de la géométrie de niveau de l'enseignement supérieur et la capacité de résoudre les équations différentielles élémentaires avec les subtilités théoriques correspondantes. Familiariser les bases de démonstrations reposant sur la conception géométrique.
Nombre minimal d'étudiants pour autoriser l'ouverture de l'UE : 5
Conditions d'attribution des crédits : La participation régulière et active à tous les cours de l'UE ainsi que la réussite aux contrôles de connaissances.
Evaluation de l'UE : Examen final écrit Une note inférieure ou égale à 05/20 (note éliminatoire) à une ECUE entraîne l'absence de note à l'UE
Intitulé des éléments constitutifs/matières constituant l'UE : 1 – Espaces affines et analyse vectorielle 2 – Intégrations et courbes paramétrées

1	**Intitulé de l'EC :** Espaces affines et analyse vectorielle
a	**Responsable de l'EC : Randriambololondrantomalala Princy**, Maître de Conférences d'ESR en activité, 0331246249, princypcpc@yahoo.fr
b	**Taille de groupe :** Cours : selon la capacité d'accueil TD : 25 par groupe
c	**Pré-requis :** *Algèbre linéaire et Analyse de base (Niveau S1), Géométrie niveau terminale S*
d	**Forme d'examen:** Epreuve écrite sur table, test oral.

e	**Compétences visées et évaluation des compétences** : *Les apprenants comprennent la conception géométrique des espaces affines et des transformations affines correspondantes. Ils sont en mesure de reconnaître la continuité, la différentiabilité des fonctions de plusieurs variables et des fonctions vectorielles. Ils peuvent aussi assimiler l'utilisation des différents opérateurs utilisés en mécanique ou physique comme la divergence, le rotationnel, le gradient, le laplacien, …etc en utilisant les dérivées partielles.*
f	**Contrôle des connaissances :** ET : Examen terminal
g	**Fréquence de l'offre :** Chaque semestre
h	**Nature des activités pédagogiques en présentiel :** C, TD.
i	**Contenu du cours et documentation pour l'étudiant :** - **Espaces affines** - **Analyse vectorielle** <u>Documentation</u> - *P. Danko et A. Popov, Exercices et problèmes des mathématiques supérieures, partie II, Editions MIR, Moscou (1977)* - *Marie-Anne Maingueneau, 30 semaines de khôlles en mathématiques, première partie et deuxième partie, ellipses (1998)* - www.mines-ales.fr/~cmgd/MMS/Cours/optapp/opvect.pdf - www.math.jussieu.fr/~delabrie/PM2/PM2.pdf - *stephane.gonnord.org/PCSI/Geom/**AFFINES**.PDF* - www.math.u-psud.fr/~pansu/websm/courbes_parametrees.pdf - *G.H. Rahaingoarivony, Exercices d'Analyse, Tranopirinty FOFIPA (1987)*
j	**Description du travail personnel de l'étudiant :** Préparation et approfondissement du cours, travail individuel et/ou en équipe, recherche de ressources documentaires/ d'informations (dans une bibliothèque, par internet …), établissement de fiches de lecture.
k	**Eléments de pédagogie (en classe) :** ***Cours*** : Enseignement magistral ***ED*** : Démonstration, exercices, apprentissage par la comparaison, exposé, étude de cas, travail en groupe ***Moyens utilisés*** : Moyens individuels, audiovisuels, polycopiés d'exercices
2	**Intitulé de l'EC :** Intégrations et courbes paramétrées.
a	**Responsable de l'EC : Ravelonirina Sammy**, Maître de Conférences d'ESR en activité, Géométrie différentielle, 0331121675, rhsammy@yahoo.fr

b	**Taille de groupe :** Cours : selon la capacité d'accueil TD : 25 par groupe
c	**Pré-requis : Mathématiques en**Terminal S, fonctions de plusieurs variables
d	**Forme d'examen:** Epreuve écrite sur table, test oral.
e	**Compétences visées et évaluation des compétences** : Les étudiants *peuvent résoudre les équations différentielles élémentaires de tout ordre par des méthodes standard respectant les vertus théoriques. Ils peuvent calculer les intégrales doubles, curvilignes et triples en connaissant les significations géométriques et mécaniques correspondantes. Les étudiants sont capables de tracer les courbes planes dans R^2 sous forme paramétrée ou polaire.*
f	**Contrôle des connaissances :** ET : Examen terminal
g	**Fréquence de l'offre :** Chaque semestre
h	**Nature des activités pédagogiques en présentiel :** C, TD.
i	**Contenu du cours et documentation pour l'étudiant :** • Intégrations • Courbes parametrées Documentation - *P. Danko et A. Popov, Exercices et problèmes des mathématiques supérieures, partie II, Editions MIR, Moscou (1977)* - *Marie-Anne Maingueneau, 30 semaines de khôlles en mathématiques, première partie et deuxième partie, ellipses (1998)* - www.mines-ales.fr/~cmgd/MMS/Cours/optapp/opvect.pdf - www.math.jussieu.fr/~delabrie/PM2/PM2.pdf - *stephane.gonnord.org/PCSI/Geom/**AFFINES**.PDF* - www.math.u-psud.fr/~pansu/websm/courbes_parametrees.pdf - *G.H. Rahaingoarivony, Exercices d'Analyse, Tranopirinty FOFIPA (1987)*
j	**Description du travail personnel de l'étudiant :** Préparation et approfondissement du cours, travail individuel et/ou en équipe, recherche de ressources documentaires/ d'informations (dans une bibliothèque, par internet …), établissement de fiches de lecture.
k	**Eléments de pédagogie (en classe) :** ***Cours*** : Enseignement magistral

	ED : Démonstration, exercices, apprentissage par la comparaison, exposé, étude de cas, travail en groupe ***Moyens utilisés*** : Moyens individuels, audiovisuels, matériels de travaux pratiques, protocoles expérimentaux, polycopiés d'exercices

e	**Compétences visées et évaluation des compétences** : A l'issue de la formation l'étudiant sera capable de : + Maîtriser les fondements de la thermodynamique et les appliquer à l'étude des types de systèmes et aux réactions chimiques + Comprendre et appliquer le processus de la cinétique de transformation des réactifs en produits lors d'une réaction chimique
f	**Contrôle des connaissances** : ET : Examen terminal + rattrapage EP : contrôle continu + examen final
g	**Fréquence de l'offre** : Tous les deux semestres
h	**Nature des activités pédagogiques en présentiel** : Cours, TD, TP
i	**Contenu du cours et documentation pour l'étudiant** : + Systèmes et milieux + Les principes de la thermodynamique (Enthalpie, entropie, énergie libre,…) + Les réactions chimiques + Equilibres chimiques (loi de l'action de masse) et physiques (états et transformations de la matière) + Cinétique chimique (Effet de la température sur la vitesse de réaction, formule d'ARRHENIUS, notion de catalyseur) <u>Documentation</u> + Robert Roux 2002. Thermodynamique Chimique cours et exercices corrigés. Dunod, Sciences Supérieur + Gruger Alain 2004, 2è édition. Thermodynamique Chimique et Equilibre chimique. Dunod + Ellipses 2006. Thermodynamique Chimique. Collection PCEM + Claude Moreau et Jean Paul Payen 1996. Thermodynamique Chimique
j	**Description du travail personnel de l'étudiant** : Préparation et approfondissement du cours, travail individuel et/ou en équipe, recherche de ressources documentaires/ d'informations (dans une bibliothèque, par internet …), établissement de fiches de lecture, rédaction de compte rendu de travaux pratiques, rédaction de rapport (stage, visite d'entreprise …), préparation d'un exposé oral, élaboration de supports de présentation.
k	**Eléments de pédagogie (en classe)** : *Cours* : Enseignement magistral *ED* : Démonstration, exercices, apprentissage par la comparaison, exposé, étude de cas, travail en groupe *EP* : Manipulation, travail en groupe *Moyens utilisés* : Moyens individuels, audiovisuels, matériels de travaux pratiques, protocoles expérimentaux, polycopiés d'exercices

Code et intitulé UE : SVTL11MATHSVT, MATHEMATIQUE POUR SVT **Type et statut de l'UE**: Obligatoire **Responsable de l'UE** : RANDRIAMBOLOLONDRANTOMALALA Princy **e-mail** : princy-pc@wanadoo.mg, **tél.** : 033 12 462 49
Charge totale du travail étudiant : 120H
Nombre de crédits : 4
Semestre d'études : Semestre 1
Compétences visées et évaluation des compétences : A l'issue de la formation l'étudiant sera capable d'utiliser les outils mathématiques et statistiques pour la compréhension des phénomènes biologiques
Objectif de l'UE : Acquérir l'essentiel des outils et des techniques mathématiques et statistiques pour interpréter certains problèmes issus de la Biologie
Nombre minimal d'étudiants pour autoriser l'ouverture de l'UE : 5
Conditions d'attribution des crédits : La participation régulière et active à tous les cours de l'UE ainsi que la réussite aux contrôles de connaissances.
Evaluation de l'UE Pour chaque ECUE, la note d'examen est égale à la moyenne des notes des contrôles continus x 2 et de la note de TP (moyenne des notes des comptes rendus). Les rapports de stage et autres activités feront l'objet d'une note à part à condition que les activités en question (stage, visite et autres) figurent parmi les activités pédagogiques indiquées dans l'UE (rubrique h dans les éléments constitutifs).

Pour le rattrapage, la note définitive est égale à la moyenne de la note d'examen écrit x 2 et de la note d'examen de TP. La règle de la compensation entre les notes des ECUE est acceptée à condition qu'aucune note d'ECUE ne soit éliminatoire. Une note inférieure ou égale à 05/20 (note éliminatoire) à une ECUE entraîne l'absence de note à l'UE	
Intitulé des éléments constitutifs/matières constituant l'UE : 1 – Mathématique 2 – Probabilité et statistique	
1	**Intitulé de l'EC** : MATHEMATIQUE
a	**Responsable de l'EC** : RANDRIAMBOLOLONDRANTOMALALA Princy
b	**Taille de groupe** : Cours : selon la capacité d'accueil TD : 25 par groupe TP : 25 par groupe
c	**Pré-requis** : Mathématiques du secondaire
d	**Forme d'examen**: Epreuve écrite sur table, épreuve orale.
e	**Compétences visées et évaluation des compétences** : A l'issue de la formation l'étudiant sera capable d'utiliser les fonctions mathématiques dans la compréhension des phénomènes biologiques
f	**Contrôle des connaissances** : ET : Examen terminal + rattrapage EP : contrôle continu sans rattrapage ou avec rattrapage selon le cas
g	**Fréquence de l'offre** : Tous les deux semestres
h	**Nature des activités pédagogiques en présentiel** : Cours et TD
i	**Contenu du cours et documentation pour l'étudiant** : *Fonctions numériques d'une variable réelle* - Rappels : domaine de définition, opérations sur les fonctions (fonctions puissances, polynomiales, rationnelles, trigonométriques, exponentielle et logarithme) - Dérivabilité. Formule de Taylor (à l'ordre 2 ou 3), développements, limites, calculs approchés, exemples et applications - Représentations graphiques - Primitives et intégrales : définitions, méthodes de calculs (changement de variables et intégration par parties) *Equations différentielles* - Définitions et exemples - Equations différentielles linéaires du premier ordre. Méthodes de résolution. Applications - Equation différentielles linéaires du second ordre à coefficients constants. Méthodes de résolution. - Applications *Fonctions à plusieurs variables* - Définitions et exemples (on se limitera à 2 variables) - Dérivées partielles, recherche d'extremum d'une fonction, gradient et applications. *Calcul matriciel et systèmes d'équations linéaires* - Définition d'une matrice, opération sur les matrices, inversion d'une matrice carrée - Applications : résolution de systèmes linéaires
j	**Description du travail personnel de l'étudiant** : Préparation et approfondissement du cours, travail individuel et/ou en équipe, recherche de ressources documentaires/ d'informations (dans une bibliothèque, par internet …), établissement de fiches de lecture, rédaction de compte rendu de travaux pratiques, rédaction de rapport (stage, visite d'entreprise …), préparation d'un exposé oral, élaboration de supports de présentation.
k	**Eléments de pédagogie (en classe)** : *Cours* : Enseignement magistral *ED* : Démonstration, exercices, apprentissage par la comparaison, exposé, étude de cas, travail en groupe *EP* : Manipulation, travail en groupe *Moyens utilisés* : Moyens individuels, audiovisuels, matériels de travaux pratiques, protocoles expérimentaux, polycopiés d'exercices
2	**Intitulé de l'EC** : PROBABILITE ET STATISTIQUE
a	**Responsable de l'EC** : RAZAFIMAHALEO Martial
b	**Taille de groupe** : Cours : selon la capacité d'accueil TD : 25 par groupe

Un petit résumé sur les maths pour les sciences vivantes et de la terre

Mention Biologie

par

Princy RANDRIAMBOLOLONDRANTOMALALA

Faculté des Sciences, Université d'Antananarivo, 2014

Chapitre 1

Fonctions et équations différentielles

1.1 Généralités

Une fonction numérique d'une variable réelle f est une application qui à $x \in \mathbb{R}$ correspond à un réel $f(x) \in \mathbb{R}$. Si x varie dans $\mathbb{R}$, alors $f(x)$ varie avec x dans $\mathbb{R}$. $f(x)$ pour x sans valeur déterminée est l'expression de la fonction f. Pourtant, il existe une partie de $\mathbb{R}$ où f est bien définie, c'est le domaine de définition de f noté D_f. C'est-à-dire que seuls les $x \in D_f$, ont des $f(x) \in \mathbb{R}$. Remarquer que dans $\mathbb{R} =]-\infty, +\infty[$, seuls les $-\infty$, $+\infty$ ne sont pas des réels. Donc, quand on cherche le D_f, on s'assure que f y est bien définie. Des petits règles de calculs sont les suivants :

• Si dans l'epression de f figure un dénominateur $\frac{h(x)}{g(x)}$, alors ce dénominateur doit être différent de 0. Donc on cherche l'ensemble des réels x tels que $g(x)$ soit non nul.

Exemple 1. Soit $f(x) = \frac{1}{x+4} + \frac{x-1}{x^2-3x+2}$, alors il faut que $x + 4 \neq 0$, $x^2 - 3x + 2 \neq 0$. C'est-à-dire que $x \neq -4$, $x \neq 1$ et $x \neq 2$. Donc $D_f =]-\infty, -4[\cup]-4, 1[\cup]1, 2[\cup]2, +\infty[$.

• Si dans l'epression de f figure une racine carrée $\sqrt{g(x)}$, alors l'expression dans la racine carrée doit être positive ou nulle. Donc on cherche l'ensemble des réels x tels que $g(x) \geq 0$.

Exemple 2. Soit $f(x) = \sqrt{x^2 - 5x + 6} + x^2$, alors il faut que $x^2 - 5x + 6 \geq 0$. C'est-à-dire que $x \leq 2$ et $x \geq 3$. Donc $D_f =]-\infty, 2] \cup [3, +\infty[$.

• Si dans l'epression de f figure un logarithme Neperien $\ln(g(x))$, alors l'expression dans le logarithme doit être strictement positif. Donc on cherche l'ensemble des réels x tels que cette expression $g(x) > 0$.

Exemple 3. Soit $f(x) = \ln(-x^2 + 5x - 6) - e^{-x}$, alors il faut que $-x^2 + 5x - 6 > 0$. C'est-à-dire que $x > 2$ et $x < 3$. Donc $D_f =]2, 3[$.

Il se peut que ces cas se présentent simultanément dans l'expression d'une fonction, alors on fait l'intersection des intervalles ainsi obtenus pour avoir le domaine de définition. Si l'expression d'une fonction ne comporte pas l'un de ces trois types d'expressions, donc son domaine de définition est $\mathbb{R}$. Des opérations définies sur les fonctions f et g est définie par $\forall x$, $(f+g)(x) = f(x) + g(x)$, $(f \times g)(x) = f(x) \times g(x)$ et $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.[1] On dit qu'une fonction est continue en un point x_0, si

$$\lim_{x \to (x_0)^-} f(x) = \lim_{x \to (x_0)^+} f(x) = f(x_0)$$

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Toute fonction usuelle est continue sur son domaine de définition.
On dit qu'une fonction est dérivable en un point x_0, si

$$\lim_{x\to(x_0)^+}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to(x_0)^-}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)$$

existe et fini (réel unique). Toute fonction usuelle est dérivable sur le plus grand ouvert contenu dans son domaine de définition. En générale, les domaines de définitions sont des reunions disjointes d'intervalles, alors pour avoir ce plus grand ouvert, il suffit de rendre chaque intervalle fermé ou semi-fermé à un intervalle ouvert correspondant.

Exemple 4. Soit $f(x)=\sqrt{-x^2+5x-6}-e^{-x}$. Donc $D_f=[2,3]$, f est dérivable sur $]2,3[$.

Après avoir faire la dérivée f' d'une fonction numérique f, on cherche les points x où $f'(x)=0$. Ce sont les points où f présente ses maximaux ou minimaux. On peut dresser son tableau de variation. Maintenant, observons les branches infinies de la graphe $\mathcal{C}$ de f.
Si

$$\lim_{x\to x_0\in\mathbb{R}} f(x)=\infty$$

alors $x=x_0$ est un asymptote vertical de $\mathcal{C}$ au voisinage de y à ∞.
Si

$$\lim_{x\to\infty} f(x)=y_0\in\mathbb{R}$$

alors $y=y_0$ est un asymptote horizontal de $\mathcal{C}$ au voisinage de ∞.
Si

$$\lim_{x\to\infty} f(x)=\infty$$

alors on fera $\lim_{x\to\infty}\frac{f(x)}{x}=a$:
Si $a=\infty$, alors l'axe des ordonnées $x=0$ est une direction asymptotique de la branche parabolique de $\mathcal{C}$ au voisinage de ∞.
Si $a=0$, alors l'axe des abscisses $y=0$ est une direction asymptotique de la branche parabolique de $\mathcal{C}$ au voisinage de ∞.
Si $a\in\mathbb{R}^*$, alors on fait $\lim_{x\to\infty} f(x)-ax=b$, donc :
• pour $b=\infty$, alors $y=ax$ est une direction asymptotique de la branche parabolique de $\mathcal{C}$ au voisinage de ∞.
• pour $b\neq\infty$, alors $y=ax+b$ est un asymptote oblique de $\mathcal{C}$ au voisinage de ∞.
Maintenant, on peut dresser le graphe de f dans un repère orthonormé $(0,\vec{i},\vec{j})$.

1.2 Développement limité

Soit f une fonction numérique d'une variable réelle, f est de classe C^n sur un voisinage de x_0, c'est-à-dire que la n-ième dérivée de f sur ce voisinage est continue. Une telle fonction admet un développement limité d'ordre n au voisinage de x_0, c'est-à-dire que f peut s'écrire approximativement en un polynôme de degré n (Formule de Taylor) :

$$f(x)=f(x_0)+f'(x_0)\frac{(x-x_0)}{1!}+f''(x_0)\frac{(x-x_0)^2}{2!}+\cdots+f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}+o((x-x_0)^n) \tag{1.1}$$

où $\lim_{x\to x_0} o((x-x_0)^n) = 0$.
Voici quelques développements limités usuels au voisinage de 0 et d'ordre n :

$$(1+x)^\alpha = \sum_{k=0}^{n} C_\alpha^k x^k + o(x^n),\ e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n),\ \ln(1+x) = \sum_{k=0}^{n} \frac{(-1)^k x^{k+1}}{k+1} + o(x^{n+1})$$

où $C_\alpha^k = \frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!}$.

$$\sin(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + o(x^{2n+1}),\ \cos(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} + o(x^{2n})$$

$$\sinh(x) = \sum_{k=0}^{n} \frac{1}{(2k+1)!} x^{2k+1} + o(x^{2n+1}),\ \cosh(x) = \sum_{k=0}^{n} \frac{1}{(2k)!} x^{2k} + o(x^{2n})$$

$$\arctan(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1} + o(x^{2n+1})$$

Le développement limité d'ordre n au $v(x_0)$ d'une somme de fonction est la somme de leurs developpements limités respectifs. Le développement limité d'ordre n au $v(x_0)$ d'un produit de 2 fonctions est le produit de leurs développements limités respectifs en ne considérant que les puissances en $(x-x_0)$ inférieures ou égales à n. Le développement limité d'une division de deux fonctions est obtenue en faisant la division d'ordre croissant selon la puissance en $(x-x_0)$, en ne considérant que les termes dont la puissance en $(x-x_0)$ est inférieure ou égale à n. Si une fonction f est définie sur un voisinage de x_0, alors $\int_{x_0}^{x} f(h)dh$ admet un développement limité d'ordre $n+1$ au $v(x_0)$ qui est l'intégrale correspondant du développement de degré n de f au $v(x_0)$. Si f admet un D.L d'ordre n au $v(x_0)$ et g en admet d'ordre n au $v(f(x_0))$, alors $g \circ f$ admet un D.L d'ordre n au $v(x_0)$ qui est la composition respective des développements limités correspondants.

1.3 Equations différentielles linéaires d'ordre $n \leq 2$

1.3.1 Equations différentielles linéaires d'ordre 1

C'est une équation de la forme

$$y' + P(x)y = Q(x) \tag{1.2}$$

avec P et Q des fonctions réelles. Alors, on cherche y et son ensemble de définition. On résoud d'abord $y' + P(x)y = 0$ et a pour solution $y_0 = Ce^{-\int P(x)dx}$ avec $C \in \mathbb{R}$. Ensuite, on fait varier C en $C(x)$ et $y = C(x)e^{-\int P(x)dx}$. En transportant cette dernière relation dans (1.2), on obtient

$$C(x) = \int Q(x)e^{\int P(x)dx}dx + C_0$$

et enfin

$$y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + C_0\right]$$

1.3.2 Equations différentielles linéaires d'ordre 2 à coefficients constants

C'est une équation de la forme

$$y'' + ay' + by = f(x) \tag{1.3}$$

On résoud d'abord l'équation homogène

$$y'' + ay' + by = 0$$

Alors, on cherche les solutions de l'équation caractéristique suivante

$$r^2 + ar + b = 0$$

$*$ Si $\Delta > 0$, alors $r_1,\ r_2$ sont deux solutions distinctes.
Donc la solution de l'équation homogène est, $y_0 = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.
$*$ Si $\Delta = 0$, alors r_1 est une solution doubles.
Alors la solution de l'équation homogène est, $y_0 = (C_1 x + C_2)\, e^{r_1 x}$.
$*$ Si $\Delta < 0$, alors $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$ sont deux solutions complexes distinctes et conjuguées.
Alors la solution de l'équation homogène est, $y_0 = e^{\alpha x} \left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$.
Pour la solution particulière y_1 de l'équation (1.3), on choisit la forme de

$$f(x) = e^{\delta x} \left(P_k(x) \cos(\gamma x) + Q_m(x) \sin(\gamma x)\right)$$

où $P_k(x)$ respectivement $Q_m(x)$ sont des polynômes de degré respectif k et m. On compte le nombre de fois où $\delta + i\gamma$ figure parmi les racines de l'équation caractéristique, on le note par j. Alors,

$$y_1 = x^j e^{\delta x} \left(G_l(x) \cos(\gamma x) + H_l(x) \sin(\gamma x)\right)$$

avec $l = max(k, m)$, $G_l(x)$ et $H_l(x)$ sont des polynômes de degré l. Les coefficients de ces derniers polynômes sont obtenus à partir du remplacement de y dans (1.3) par ce y_1.
D'où la solution finale $y = y_0 + y_1$.

1.3.3 Remarques

Les dernières constantes indéterminées de la solution finale d'une équation différentielle peuvent être déterminées en attribuant cette solution des valeurs précises en des points. Par exemple, pour l'équation différentielle de degré 1, $y(0) = 1$; pour l'équation différentielle de degré 2, $y(1) = 0, y'(0) = 3$; selon les données sur l'équation différentielle.
Si le deuxième membre de (1.3) s'écrit sous-forme de somme de deux fonctions $f(x) = f_1(x) + f_2(x)$, alors on peut chercher séparément la solution particulière de :

$$y'' + ay' + by = f_1(x) \tag{1.4}$$

qui est y_2 et,

$$y'' + ay' + by = f_2(x) \tag{1.5}$$

qui est y_3. Donc, la solution particulière de (1.3) est $y_2 + y_3$. Ainsi, la solution finale est $y = y_0 + y_2 + y_3$.

1.4 Quelques notions sur les fonctions à plusieures variables

On se limitera à une fonction numérique à 2 variables. Une telle fonction f est définie dans un domaine de $\mathbb{R}^2$, la manière de chercher le domaine de définition de f est la même que celle d'une fonction numérique d'une variable réelle, sauf que le domaine de définition est inclus dans $\mathbb{R}^2$.

Exemple 5. $f(x, y) = \ln(x + y)$, alors $D_f = \{(x, y)/x + y > 0\}$.

Une fonction usuelle est toujours continue sur son domaine de définition, est toujours différentiable (dérivable) sur le plus grand ouvert contenu dans son domaine de définition. Alors, on peut faire une dérivée partielle par rapport à x notée $\frac{\partial f}{\partial x}$ (on dérive f par rapport à x où on considère y comme une constante) ou par rapport à y $\frac{\partial f}{\partial y}$ (on dérive f par rapport à y où on considère x comme une constante).

Exemple 6. $f(x, y) = \ln(xy^2 + y)$ où $D_f = \{(x, y)/xy^2 + y > 0\}$, alors $\frac{\partial f}{\partial x} = \frac{y^2}{xy^2+y}$ et $\frac{\partial f}{\partial y} = \frac{2yx+1}{xy^2+y}$.

Théorème 7. *Toute fonction dont toutes dérivées partielles premières sont continues dans un domaine de $\mathbb{R}^2$ est différentiable (dérivable) dans ce domaine.*

Chapitre 2

Système d'équations linéaires et matrices

Une matrice réelle A du type (m, n) est contituée par un ensemble ordonné de n vecteurs à m composantes réelles, c'est-à-dire

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & & & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & & & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ & & \dots & & & \dots & \\ & & \dots & & & \dots & \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & & & \dots & a_{mn} \end{pmatrix}$$

, elle représente une application $\mathbb{R}$-linéaire de $\mathbb{R}^n$ vers $\mathbb{R}^m$. Deux matrices sont égales si chaque composantes des ces deux matrices sont égales. Si $n = m$, alors A est une matrice carrée d'ordre n. La somme de deux matrices ne se fait que si ces deux matrices sont de même type, cette somme se fait composante par composante. La multiplication $C = (c_{ij}) = AB$ de deux matrices $A = (a_{ij})$ et $B = (b_{ij})$ ne se fait que si le nombre de colonnes de A est égal au nombre de ligne de B (par exemple p), où $c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$. Il existe une matrice carrée neutre d'ordre n, I_n telle que pour toute matrice carrée A d'ordre n, $I_n A = A I_n = A$. Une matrice carrée A d'ordre n est inversible s'il existe une matrice A^{-1} inverse de A telle que $A^{-1}A = AA^{-1} = I_n$.

2.1 Calcul du déterminant d'une matrice carrée

Soit A une matrice carrée d'ordre n, on calcul le $\det(A)$:
Si A est d'ordre 1, alors $\det(A) = A$.[1]
Si A est d'ordre 2,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det(A) = ad - cd.$$

Si

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix}$$

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$\det(A)$ se calcul par la méthode de Sarrus :

$$\begin{vmatrix} a^+ & b^+ & c^+ & a & b \\ d & e & f & d & e \\ g_- & h_- & l_- & g & h \end{vmatrix}$$

$\det(A) = +(ael + bfg + cdh) - (gec + hfa + ldb)$. On peut aussi utiliser la méthode des cofacteurs, on écrit

$$A = \begin{pmatrix} a^+ & b^- & c^+ \\ d^- & e^+ & f^- \\ g^+ & h^- & l^+ \end{pmatrix},$$

les règles de signes sur chaque élément de cette matrice est $(-1)^{i+j}$ où l'élément en question est à la i-ième ligne et j-ième colonne. On choisit alors une ligne ou une colonne pour le développement du déterminant, par exemple par la première colonne. Ainsi, $\det(A) = +a\det(A_{11}) - d\det(A_{21}) + g\det(A_{31})$ où

$$A_{11} = \begin{pmatrix} e & f \\ h & l \end{pmatrix},\ A_{21} = \begin{pmatrix} b & c \\ h & l \end{pmatrix},\ A_{31} = \begin{pmatrix} b & c \\ e & f \end{pmatrix}$$

Le cofacteur de la i-ième ligne j-ième colonne est $(-1)^{i+j}\det(A_{ij})$. En général, si A est une matrice carrée d'ordre quelconque, la méthode des cofacteurs est toujours appliquable, contrairement à la méthode de Sarrus qui est seulement appliquable pour une matrice d'ordre 3.
Une matrice carrée est inversible si et seulement si elle admet un déterminant non nul. Ainsi, pour une matrice d'ordre n inversible,

$$A^{-1} = \frac{\tilde{A}^t}{\det(A)}$$

où $\tilde{A} = ((-1)^{i+j}\det(A_{ij}))$ et l'opération ^{t} (transposé) désigne le remplacement de chaque i-ième colonne d'une matrice à la i-ième ligne. On a $\det(AB) = \det(A)\det(B)$ si A et B sont des matrices carrées.

2.2 Système d'équations linéaires

un système d'équations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{2.1}$$

où chaque b_i sont des réels. On forme l'expression matricielle du système ci-dessus :

$$A \times X = B \Rightarrow \begin{pmatrix} a_{11} & a_{12} & \dots & & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots\ \vdots & & \vdots \\ & & \dots & & \dots & \\ & & \dots & & \dots & \\ \vdots & \vdots & & \vdots\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_{m-2} \\ b_{m-1} \\ b_m \end{pmatrix}$$

où

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & & & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ & & \cdots & & & \cdots & \\ & & \cdots & & & \cdots & \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & & & \cdots & a_{mn} \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix}, \; B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_{m-2} \\ b_{m-1} \\ b_m \end{pmatrix}$$

Le problème c'est de chercher $x_1, \ldots, x_n$ vérifiants à la fois chaque équation du système. La méthode de pivot de Gauss consiste à rendre cette matrice A en une matrice réduite échélonnée. On choisit une ligne appelée pivot 1, on choisit une variable à éliminer dans les autres lignes par la variable correspondante dans le pivot. Cette élimination est une suite de calcul : on peut additionner à une ligne (fois une constante) (dont la variable est à éliminer) autre que le pivot par le pivot fois une constante, de façon que la variable en question disparait dans le calcul. Lorsque cet étape est fait, on choisit un autre pivot 2 et on choisit une autre variable à éliminer dans les autres lignes différentes de pivot 1, et on refait le même processus de calcul. A la fin du processus (il n'y a plus de variable à éliminer), on obtient cette nouvelle matrice réduite, et puis la valeur des variables restantes, on remonte les valeurs de ces variables au pivot précédent et ainsi de suite. Alors on résoud le système linéaire de départ.
Si $n = m$ et que $\det(A) \neq 0$, alors la méthode de Cramer est appliquable, chaque

$$x_i = \frac{\Delta_{x_i}}{\Delta}$$

où $\Delta = \det(A)$ et $\Delta_{x_i} = \det(A_i)$ avec A_i est la matrice A où on a seulement remplacé la i-ième colonne par

$$\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{matrix}$$

Dans ce dernier cas, si $n = m$ et $\det(A) \neq 0$, on connait A^{-1} ; alors

$$X = A^{-1}B$$

et les x_i sont identifiés par cette égalité matricielle.
Ainsi, on résoud le système d'équations linéaires. [2]

2. Princy RANDRIAMBOLOLONDRANTOMALALA, Maths pour SVT, Univ Antananarivo 2014

Un petit résumé sur l'optimisation linéaire

par

Princy RANDRIAMBOLOLONDRANTOMALALA

IT University, 2014

Table des matières

Introduction

Le problème de l'optimisation consiste à chercher la valeur maximale ou minimale d'une fonction en présence de certaines contraintes. Dans ce cours, on s'intéresse sur le cas d'une fonction linéaire sous des contraintes linéaires. Il a une petite distinction sur la méthode à suivre dans cette étude. En ce qui concerne le cas de $\mathbb{R}$ ou $\mathbb{R}^2$, il suffit de faire une résolution par le dérivé d'une fonction réelle ou par la méthode graphique. Quant au cas de $\mathbb{R}^n$ avec $n \geq 3$, la méthode appropriée c'est une méthode analytique, la méthode du simplexe. Bien sûre, cette dernière méthode est aussi valable pour la résolution de ces premiers cas. [1]

1. Princy RANDRIAMBOLOLONDRANTOMALALA, Optimisation

Chapitre 1

Programmation linéaire en deux dimension

Le but c'est de chercher la valeur optimale de la fonction linéaire $L = ax + by$ avec $a, b \in \mathbb{R}$, (x, y) le système de coordonnées dans $\mathbb{R}^2$ sous les contraintes

$$\begin{cases} a_1x + b_1y \geq C_1 \\ a_2x + b_2y \geq C_2 \\ \dots \\ a_nx + b_ny \geq C_n \end{cases}$$

Une méthode usité est la méthode par graphique, il s'agit de dessiner sur $\mathbb{R}^2$ rapporté à un repère orthonormé $(O, \vec{i}, \vec{j})$ toutes les droites correspondantes à ces contraintes, hachurer chaque partie qui ne contient pas la solution. Une droite de $\mathbb{R}^2$ a toujours deux cotés gauche et droite. Alors si un point appartenant à sa gauche et n'appartenant pas à la droite ne vérifie pas l'inéquation correspondante alors c'est la partie droite qui est la solution, et on doit hachurer la partie gauche ; et vice versa. En effectuant ce processus sur toute n les droites définies par les contraintes, on obtient une partie non hachurée de $\mathbb{R}^2$ qui est le domaine de solution de notre problème. On trace maintenant le vecteur partant de l'origine O de composante $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$. Alors on fait glisser une perpendiculaire à la droite dans le domaine de solution. Si on cherche le maximum de L, alors l'ensemble de points frontières commune à cette normale glissée dans le sens du vecteur $\vec{u}$ avec le domaine de solution constitue les points où L est maximale. Avec le même principe mais dans le sens contraire de $\vec{u}$, on obtiendra le minimum de L.

Remarque 1. On peut toujours ramener une inéquation comportant $\leq$ à $\geq$ dans les contraintes ci-desssus en multipliant chaque membre par un $-$.

[1]

Remarque 2. Pour un domaine D de solution bornée, il suffit de trouver les valeurs de L pour chaque sommet en nombre fini, et les comparer ensuite. Pour ce faire, chaque sommet est la solution du système de deux équations à deux inconnues correspondants aux deux droites définies par les contraintes deux à deux.

1. Princy RANDRIAMBOLOLONDRANTOMALALA, Optimisation

Exercice 3. Maximisation et minimisation de la forme linéaire $L = x+y$ avec les contraintes

$$\begin{cases} 3x - 2y \geq -6 \\ -3x - y \leq -3 \\ x \leq 4 \end{cases}$$

Solution : $L_{max} = 13$ au point $(4, 9)$, $L_{min} = -5.00$ au point $(4, -8.999)$.

Exercice 4. Maximisation et minimisation de la forme linéaire $L = x + 2y$ avec les contraintes

$$\begin{cases} 2x - 3y \leq 0 \\ x + y \geq 1 \end{cases}$$

Solution : $L_{max} = +\infty$, $L_{min} = 1.400$ au point $(0.599, 0.400)$.

Exercice 5. Maximisation de la forme linéaire $L = 4x - y$ avec les contraintes

$$\begin{cases} 3x + 4y \geq 18 \\ 3x - y \geq 3 \\ y \leq 6 \\ 2x + y \leq 18 \\ 4x - y \leq 24 \end{cases}$$

Solution : $L_{max} = 24$, aux points situés dans le domaine de solutions de la droite $y = 4x - 24$.

Cependant, il est fort possible que la forme linéaire, les contraintes ne sont pas explicitées mais on les formule partant des données de l'exercice. Dans ce cas il faut bien interpréter mathématiquement les données du problème. Il faut en premier placer les variables en question, souvent indiquées par les données sur la fonction linéaire à optimiser. Deuxiemement, écrire la fonction linéaire à optimiser et troisièmement les contraintes du problème.

Exercice 6. Une entreprise fabrique des postes de télé et des ordinateurs. 140 ouvriers travaillent à la fabrication. Le prix de revient, pièces et main d'oeuvre d'un ordinateur est de 400 000 Ariary. Il n'est que de 300 000 Ariary pour un poste de TV.
On ne peut pas dépasser par semaine la somme de 240 000 000 Ariary, pièces et main d'oeuvre et chaque ouvrier travaille 40 heures par semaine.
Les gérants de l'entreprise estiment qu'il faut 10 heures de main d'oeuvre pour fabriquer un poste de TV et 5 heures seulement pour fabriquer un ordinateur. L'entreprise ne peut vendre plus de 480 ordinateurs et 480 poste de TV par semaine. Les prix de ventes sont tels que l'entreprise, tous frais payés, fait un bénéfice de 240 000 Ariary par ordinateur et de 160 000 Ariary par postes de TV.
1) Ecrire le bénéfice.
2) Chercher les contraintes correspondantes à ce problème.
3) Déterminer la fabrication qui assure un bénéfice maximum.

Pour résoudre le problème, on sait que le bénéfice B est la fonction à maximiser. Or le bénéfice dépend du nombre d'ordinateur et de TV vendus. Alors, les variables sont x et y qui représentent respectivement le nombre d'ordinateurs et de TV vendus. Ainsi, $B = 240000\ Ariary\ x + 160000\ Ariary\ y$. Evidemment, ces nombres sont positifs ou nuls.

Maintenant, on interprète les autres données pour former les autres contraintes. Il est immédiat que $x \leq 480$ et $y \leq 480$ d'après les dernières données. Concernant les prix de revient, on a $400000\ Ariary\ x + 300000\ Ariary\ y \leq 240000000 Ariary$. Pour les heures de travail des ouvriers $10\ h\ y + 5\ h\ x \leq 140 \times 40\ h$. Maintenant, il faut choisir une echelle appropriée pour la résolution graphique du problème. Par exemple, on prend une unité de 1 000 pour les axes des x et des y. Ainsi, $B_{max} = 140800000\ Ariary$.

Chapitre 2

Programmation linéaire en dimension $n \geq 2$

On essaie de chercher par la méthode de simplexe la valeur optimale d'une fonction linéaire $L = c_1x_1 + \cdots + c_nx_n + c_0$ parmi les solutions positives ou nulles avec les contraintes exprimées par un système d'équations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{2.1}$$

où chaque b_i sont positifs ou nuls. On forme l'expression matricielle des contraintes ci-dessus :

$$A \times X = B \Rightarrow \begin{pmatrix} a_{11} & a_{12} & \dots & & & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & & & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ & & \dots & & & \dots & \\ & & \dots & & & \dots & \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & & & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_{m-2} \\ b_{m-1} \\ b_m \end{pmatrix}$$

On cherche le rang de la matrice A, supposons que c'est $r \geq 1$. Alors, pour simplifier l'écriture, on assume que $x_1, \dots, x_r$ sont les variables de base et $x_{r+1}, \dots, x_n$ les variables libres. On change l'expression de L en fonction des variables libres x_i $r+1 \leq i \leq n$ et on transforme les coefficients des x_i $1 \leq i \leq r$ dans (2.1) en 1 chacun. Alors, on a

$$\begin{cases} x_1 + \cdots + c_{1r+1}x_{r+1} + \cdots + c_{1n}x_n = e_1 \\ x_2 + \cdots + c_{2r+1}x_{r+1} + \cdots + c_{2n}x_n = e_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_r + \cdots + c_{rr+1}x_{r+1} + \cdots + c_{rn}x_n = e_r \end{cases} \tag{2.2}$$

où chaque b_i sont positifs ou nuls. $L + d_{r+1}x_{r+1} + \cdots + d_n x_n = d_0$ et on peut dresser le tableau suivant :

$V.B$	$T.C$	x_1	...	...	x_i	...	x_r	x_{r+1}	...	x_j	...	x_n
x_1	e_1	1	0	...	0	...	0	c_{1r+1}	...	c_{1j}	...	c_{1n}
...	...	...	...	...	...	...	...	...	...	...	...	...
x_i	e_i	0	0	...	1	...	0	c_{ir+1}	...	c_{ij}	...	c_{in}
...	...	...	...	...	...	...	...	...	...	...	...	...
x_r	e_r	0	0	...	0	...	1	c_{rr+1}	...	c_{rj}	...	c_{rn}
L	d_0	0	0	...	0	...	0	d_{r+1}	...	d_j	...	d_n

où $V.B$ désigne la colonne des variables de bases, $T.C$ celle des termes constants. Les d_j sont appelés estimations des x_j correspondants.
Le choix de la colonne résolvante p est de façon que $d_p < 0$ et il existe $c_{ip} > 0$. La ligne résolvante est obtenue par q où $\frac{e_q}{c_{qp}}$ minimum parmi les $\frac{e_i}{c_{ip}}$ avec $c_{ip} > 0$. Après, on calcul à nouveau la q-ième ligne résolvante par $c'_{qk} = \frac{c_{qk}}{c_{qp}}$. Les autres lignes deviennent $c'_{ik} = c_{ik} - c'_{qk}c_{ip}$ où $k \neq p$ et $i \neq q$.

2.1 Pour un problème non dégénéré

Théorème 7. *Lors d'une itération, s'il existe une estimation négative stricte et un élément de la colonne correspondante est positif strict, alors on peut améliorer la solution par l'itération suivante. S'il existe une estimation négative dont la colonne correspondante n'a pas d'élément positif, alors $L_{max} = \infty$. Si toutes les estimations sont non négatives, alors L_{max} est obtenue.*

Exemple 8. Trouver le maximum de L avec $L = 5x_1 + 7x_2$ où les $x_i \geq 0$ et $3x_1 + 2x_2 + x_3 = 19$, $x_1 + 2x_2 + x_4 = 13$, $3x_1 + x_5 = 15$ et $3x_2 + x_6 = 18$. Alors $L_{max} = 50$.

2.2 Pour un problème dégénéré

C'est le cas où dans une itération, on trouve $e_i = 0$. Dans ce cas, il faut choisir la ligne résolvante dans laquelle le rapport minimal des éléments de la colonne suivante de la colonne résolvante aux éléments correspondants à la colonne résolvante. Et ainsi de suite jusqu'à avoir la détermination univoque de la ligne résolvante.

Exemple 9. Trouver le maximum de $L = 2x_6 + 4x_5$ où les $x_i \geq 0$ avec $x_1 + x_6 + x_5 = 12$, $x_2 - x_6 + 5x_5 = 30$, $x_3 + x_5 - 2x_6 = 6$, $2x_4 + 3x_5 - 2x_6 = 18$; alors $L_{max} = 38$.

Chapitre 3

Méthode duale

Il s'agit de représenter le problème de départ appelé problème primal sous-forme d'un autre problème équivalent appelé problème dual et vice versa. On suppose que toutes les variables sont non négatives. Si on minimise $L = c_1x_1 + \cdots + c_nx_n$ où

$$
\begin{array}{cccccccl}
a_{11} & a_{12} & \ldots & & & \ldots & a_{1n} & \geq b_1 \\
a_{21} & a_{22} & \ldots & & & \ldots & a_{2n} & \geq b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \\
 & & \ldots & & & \ldots & & \\
 & & \ldots & & & \ldots & & \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \\
a_{m1} & a_{m2} & \ldots & & & \ldots & a_{mn} & \geq b_m
\end{array}
$$

alors le problème dual équivalent est le suivant, on maximise $L' = b_1y_1 + \cdots + b_my_m$ sous les contraintes

$$
\begin{array}{cccccccl}
a_{11} & a_{21} & \ldots & & & \ldots & a_{m1} & \leq c_1 \\
a_{12} & a_{22} & \ldots & & & \ldots & a_{m2} & \leq c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \\
 & & \ldots & & & \ldots & & \\
 & & \ldots & & & \ldots & & \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \\
a_{1n} & a_{2n} & \ldots & & & \ldots & a_{mn} & \leq c_n
\end{array}
$$

Dans ce cas $L_{min} = L'_{max}$. On remarque que la matrice correspondante au problème dual est la transposée de la matrice correspondante au problème primal et vice versa.

Chapitre 4

Problèmes de transport

Le problème de transport que nous allons traité ici c'est l'organisation la plus rationnelle des transports des charges avec un minimum de coûts. Supposons qu'en p points dépôts, il y ait respectivement $a_1, \dots, a_p$ unités de charge homogène qui'il faut transporter à q utilisateurs devant recevoir respectivement $b_1, \dots, b_q$ unités. Les frais c_{ik} qu'il faut engager pour le transport d'une charge unitaire du i-ième point d'expédition au k-ième utilisateur. La variable x_{ik} désigne la quantité positive de nombre d'unités de charges du i-dépôt vers k-utilisateur. Alors pour satisfaire les demandes, on doit avoir

$$\sum_{k=1}^{q} x_{ik} = a_i, \ \sum_{i=1}^{p} x_{ik} = b_k, \ x_{ik} \geq 0.$$

Les frais de transport est $L = c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{pq}x_{pq}$. Donc, on minimise L avec les contraintes ci-dessus.
Pour ce faire, on passe par deux étapes, c'est 1) la détermination de la solution de base initiale (méthode de coin "Nord-west") et 2) la construction des itérations successives pour avoir le minimum de L (méthode des potentiels). On forme le tableau suivant :

a_i/b_k	b_1	$\dots$	b_k	$\dots$	b_q
a_1	${x_{11}}^{c_{11}}$	$\dots$	${x_{1k}}^{c_{1k}}$	$\dots$	${x_{1q}}^{c_{1q}}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a_i	${x_{i1}}^{c_{i1}}$	$\dots$	${x_{ik}}^{c_{ik}}$	$\dots$	${x_{iq}}^{c_{iq}}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a_p	${x_{p1}}^{c_{p1}}$	$\dots$	${x_{pk}}^{c_{pk}}$	$\dots$	${x_{pq}}^{c_{pq}}$

4.1 Solution de base initiale

On remplit le tableau ci-dessus en commencant par remplir le coin extrême gauche du haut (Nord-West). C'est à dire satisfaire les besoins b_1 par a_1, alors on met dans cette case $min(a_1, b_1)$. Si b_1 n'est pas satisfait ($a_1 < b_1$), alors on se déplace suivant cette colonne dans la case de a_2 pour satisfaire le reste qu'on a manqué dans a_1, on place dans cette case $min(a_2, b_1 - a_1)$ et ainsi de suite. Si b_1 est satisfaite, alors on cherche à satisfaire b_2 par le reste de a_1 utilisé par b_1, c'est-à-dire qu'on place dans cette case (1,2) le $min(a_1 - b_1, b_2)$, et ainsi de suite. A la fin, lorsque toutes les b_k sont satisfaites, alors on a une solution de base et on peut en déduire L dans à partir de cette solution. On la note $\gamma_0 = \sum_{i,j} c_{ij}x_{ij}$.

4.2 Méthode des potentiels

On reconnait les variables de base x_{kl} qui correspondent à la case remplie de nombre non nul, les autres variables x_{mn} sont des variables libres. Alors, on met L sous la forme $L = \sum_{m,n} \gamma_{mn} x_{mn} + \gamma_0$. On cherche γ_{mn} en assignant à chaque point dépôt a_i une valeur u_i (potentiel) et l'utilisateur b_j une valeur v_j (potentiel) de façon que

$$u_k + v_l = c_{kl}$$

en donnant à l'une de ces valeurs (potentiels) une valeur précise. On désigne par

$$c'mn = u_m + v_n.$$

Alors,

$$\gamma_{mn} = c_{mn} - c'mn.$$

Si toutes les γ_{mn} sont non négatives, la solution γ_0 est le minimum. Mais s'il existe une qui est négative, on passe à la base suivante en augmentant la variable à coefficient négatif choisie et en laissant les autres variables libres à zéro.

Exemple 10. En deux dépôts A et B sont stockés respectivement 1500 et 900 t de carburants. Les demandeurs 1, 2, 3 ont besoin respectivement de 600, 700 et 1100 t. Les coûts de transports d'une tonne de carburants de A vers 1, 2 et 3 sont respectivement de 60, 100 et 40 euros, mais de B vers 1, 2 et 3 sont respectivement de 120, 20 et 80 euros. Chercher l'organisation du transport de manière à minimiser les frais. (Frais minimum=102 000 euros).

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